// THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-Office at Kidder, Missouri, as Second-Class Mail Matter.

VOL. II. SEPTEMBER-OCTOBER, 1895. No. 9-10.

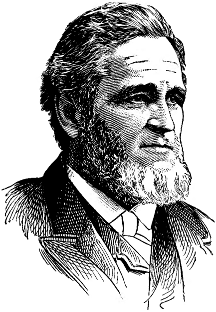
BIOGRAPHY.

PROFESSOR DE VOLSON WOOD.

BY F. P. MATZ.

DE VOLSON WOOD was born near Smyrna, New York, in 1832. To 1852 he enjoyed the educational advantages of the common school, of six weeks in a private academy, and of one term in Cazenovia Seminary. He began teaching public school, in 1849; and in his native town, Smyrna, he taught three terms. In 1853, he was graduated by the Albany (State) Normal School; and during 1853 and '54, he was the Principal of Schools, Napanoch, Ulster county. He was Assistant Professor of Mathematics in the Albany Normal School, 1854-1855; Assistant Teacher and Student, Rensselaer Polytechnic Institute, Troy, 1855-1857; from which Institution he was graduated with the Degree of Civil Engineer (*C. E.*), in 1857; was *honored* by Hamilton College, in 1859, with the Degree *A. M.* He was a Professor in the University of Michigan, from 1857 to 1872; and from this University, he received the Degree of *M. Sc.*, in 1859. He served as Professor of Mathematics and Mechanics in Stevens Institute of Technology, Hoboken, New Jersey, from 1872 to 1885; and in the capacity of Professor of Mechanical Engineering, he is serving this Institute, since 1885.

Professor Wood was a member of the American Society of Civil Engineers, from 1871 to 1885. He has been a member of the American Association for the Advancement of Science, since 1879; and he was the Vice President of this Association, in 1885. Professor Wood is a member of the American Mathematical Society, and an *honorary* member of the American Society of Architects. He has been a member of the American Society of Mechanical Engineers, since 1886; was the *first* President of the Society for the Promotion of Engineering Education; and was the Engineer of the Ore-Dock, Marquette,



DeVolson Wood.

Michigan, in 1864. He is the *inventor* of "Wood's Steam Rock-Drill," 1866 and later; and he is, also, the inventor of other machinery.

Among the articles contributed by Professor Wood, to various magazines, books, etc , may be mentioned: *Alligation*, to the "New York Teacher" —and highly commended in "Brooks's History of Arithmetic;" *Foundations*, in "Johnson's Cyclopedia;" *Mechanics*,in Appleton's Cyclopedia of Mechanics;" *Luminiferous Aether,* in the "London Philosophical Magazine"—and in Van Nostrand's Science Series, No. 85; and *Radiant Heat not an Exception to the Second Law of Thermodynamics*, in the "American Engineer."

Professor Wood has contributed to the "AMERICAN MATHEMATICAL MONTHLY," to the "Michigan Journal of Education," to the "Journal of the Franklin Institute," to the "Railroad Gazette," to the "Mining and Engineering Journal," to the "National Educator," to the "Mathematical Visitor," to the "Analyist," to "Van Nostrand's Engineering Magazine," to the "Educational Notes and Queries," to the "American Engineer," to "Science," to the "Annals of Mathematics," to the "New England Journal of Education," to the "Mathematical Magazine," to the "Engineer," to the "Barnes Educational Monthly," to the "Mathematical Messenger," etc., etc.

Professor Wood is the *author* of the following books: *Trusses, Bridges and Roofs,* published in 1872; *Wood's Edition of Mahan's Civil Engineering*, published in 1873; *Treatise on the Resistance of Materials*, published in 1875; *The Elements of Analytical Mechanics*, published in 1876; *Wood's Edition of Magnus' Lessons in Elementary Mechanics*, published in 1878; *Co-ordinate Geometry and Quaternions*, published in 1879; *Key and Supplement to the Elements of Mechanics*, and *Key and Supplement to the Mechanics of Fluids*, both published in 1884; *Trigonometry*, published in 1885; *Thermodynamics*, published in 1887 and enlarged in 1889; and *Turbines*, published in 1895.

Professor Wood was born, and raised, on a farm. In fact, until he went to the Albany Normal School, in 1852, the farm was his home. He began teaching at the age of seventeen; *paid by teaching*, the expenses of his education, and has been teaching every year since. The only position Professor Wood ever *sought*, was the first one he ever held; and the income during the three months was thirty dollars + the *privilege* of doing all the work from Principal to Janitor + the *obligation* of boarding around.

After he was a graduate of the Rensselaer Polytechnic Institute, he "wended westward" his way—not knowing whither he was going; but while traveling through Michigan, something prompted him to visit the State University. After arriving at Ann Arbor, he called at the office of the President, Dr. H. P. Tappan, who (after *the manner* of these Dignitaries) inquired about his aims and qualifications, and then asked him to teach a few days—until they heard from a recent appointee. He began teaching, that day; and, also, remained fifteen years. During this time, Professor Wood *organized* the Department of Civil Engineering; and this Department has existed since that time. Soon after he began teaching in the Michigan State University, his funds were exhausted; and one day he declared he would write home, if he had five cents;

and just then, at the door of the University Building, he saw a *dime* lying in the sand; quickly he picked it up, and wrote home at once. While Professor Wood was teaching at Napanoch, he was granted a vacation of one week; and during this week, he attended the closing exercises at the Albany Normal School. As he entered the Principal's office, the Principal greeted him thus: "Ah! I was just writing to you and offering you the Assistant Professorship of Mathematics. Will you take it?" The offer of the Assistant Professorship of Mathematics was promptly accepted; and at the opening of the next scholastic year, Professor Wood was a member of the Faculty of the Albany Normal School.

The neighbors used to say: "The stones on Mr. Wood's farm are *covered with figures* which his son, De Volson, had used in the solutions of problems."

Possibly the greatest satisfaction to Professor Wood is the pleasure and success he has had in the class room. Men, years after graduation, have complimented him on his success. They have asked for the *secret* of this success. They have asked him to tell how he inspired with labor—and why students would, in *many* cases, put twice the labor on *his* subjects rather than on the subjects of others—and why he did not *scold* his students—and why he was universally respected by his students, etc.

Brush (of Electric fame), Cleveland, Ohio, says: "Professor De Volson Wood got more genuine study out of me than any other teacher I ever was under."

The civil, mechanical, and electrical, *engineers*, *architects*, railroad *managers* and *presidents*, college *professors* and *presidents*, etc., who formerly were Professor Wood's students and who now are scattered over the whole world, would, if simultaneously "rounded up," form the most intelligent army that ever moved on the face of this mundane sphere.

Some years ago, Professor Wood went to New Mexico—he, also, visited Gunnison, Colorado; and during his visit, he was in only one place in which he could not immediately have been identified at a bank, by one of his former pupils. Stepping off the train at Topeka, Kansas, on his return, he met a former student who had been a passenger on the same train. To him Professor Wood expressed the desire of having cashed a fifty-dollar check. The former student quickly stepped to the ticket-agent, requested him to cash the check, and Professor Wood promptly received the desired fifty dollars.

The books written by Professor Wood have proved of great assistance to science, although no *radical* reforms or changes in them are attempted. Professor Wood has unceasingly sought to make the books written by himself—*his own*; and to that extent, he made them *original*. His books have found desirable places in foreign lands. He is now engaged in *enlarging* his work on Turbines; and he hopes to make it not only rigidly theoretical, but also as practical as the solution of so difficult a problem can be made by a finite mind *working* with its own products—*under the guidance of common sense and reason.*

Professor Wood was married to Miss Cordera E. Crawe, Earlville, New York, in 1859. She died in 1866; and two years after her death, he married

Miss Fannie Hartson, Mexico, New York. By the first marriage, one son was born to them, who died in 1889. The second marriage was blessed with six children, five of whom are living. Professor Wood is a member of the Methodist Church—and, also, a member of the Official Board of the Methodist Church.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton), Ph. D., (Johns Hopkins), Member of the London Mathematical Society, and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the July-August Number.]

SCHOLION I. *In which is weighed the attempt of Proclus.* After the theorems so far demonstrated by me, independently of the Euclidean postulate, toward an exact demonstration of which they should all conspire; in my judgment it is well if I diligently weigh the labors of certain well-known geometers in the same endeavor.

I begin from Proclus, of whom Clavius in the Elements after P. XXVIII, Book I, gives the following assumption:

If from one point two straight lines making an angle are produced infinitely, their distance will exceed every finite magnitude.

But Proclus demonstrates indeed (as Clavius there well remarks) that two straights (fig. 20) as suppose AH, AD going out from the same point A toward the same parts, always diverge the more from each other, the greater the distance from the point A, but not also that this distance increases beyond every finite limit that may be designated, as was requisite for his purpose.

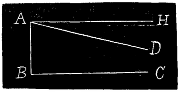


FIG. 20.

In which place the aforesaid Clavius cites the example of the Conchoid of Nikomedes, which going out from the same point A as the straight AH toward the same parts, so recedes always more from it, that nevertheless only at an infinite production is their distance equal to a certain finite sect AB standing perpendicular to AH and BC produced to infinity toward the same parts.

Why may not the same be said of the two supposed straight lines AH, AD, unless a special reason constrains to the contrary?

Nor here can Clavius be blamed that he opposes to Proklos this property of the Conchoid, which cannot be demonstrated except with the aid of many theorems resting upon the here controverted postulate.

For I say from this itself the force of the Clavian rebuttal is confirmed; for it is certain from this postulate being assumed that truly it follows manifestly, that two lines protracted to infinity, one straight, and the other

curved, can recede one from the other ever more within a certain finite determinate limit; whence at any rate may arise a suspicion lest the same may be able to happen for two straight lines, unless otherwise demonstrated.

But not therefore, after I in the corollary to the preceding proposition I have made manifest the absolute truth of the aforesaid assumption, is it possible immediately to go over to the assertion of the Euclidean postulate. For previously must also be demonstrated, that those two straights AH, BC, which with the transversal AB make two angles toward the same parts equal to two right angles, as for example each a right angle, do not also, protracted toward these parts to infinity, always separate more from one another beyond all finite assignable distance. For if one chooses to presume the affirmative, which is indeed entirely true in the hypothesis of acute angle; it certainly will not be a legitimate consequence, that the straight AD in any way cutting the angle HAB, hence of course making at the same time two internal angles DAB, CBA toward the same parts less than two right angles; that, I say, this straight AD, produced to infinity must at length meet with BC produced; even if it were at another time demonstrated, that the distance of the two AH, AD produced to infinity ever greater goes out beyond all finite limit that may be assigned.

But that the aforesaid Clavius should have judged the truth of this assumption sufficient for demonstrating the postulate here in question; that ought to becondoned because of the opinion preconceived by Clavius about equidistant straight lines, which we may discuss more conveniently in a subsequent Scholion.

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

(Continued from the July-August Number.)

Construction of Intransitive Groups.

Suppose that we have an intransitive group (G) involving the six letters a, b, c, d, e, f and that in this group a is replaced by b, c, and d but not by e or f. This group must have at least one substitution (s_1) in which a is replaced by b, one (s_2) in which a is replaced by c and one (s_3) in which a is replaced by d. In some power of s_1 (which, from the definition of a group, must also be in G) b is replaced by a.* Let this substitution be denoted by s'_1 and consider the following substitution of G:

$$s'_1,\quad s'_1\ s_2,\quad s'_1\ s_3.$$

In the first of these b is replaced by a, in the second by c and in the third by d. Hence we see that the hypothesis that a is replaced by each of the

*Suppose s_1 were one of the following substitutions: ab, abc, $abcd$, $ab.cd$; then b would be replaced by a in the first power of the first and last substitutions, in the second power of the second substitution and the third power of the third.

other letters in the first set of four requires that the other letter of this set or system have the same property, for what we proved in regard to b can be proved, similarly, in regard to c and d.

It is further evident that there can be no substitution in G which replaces one of these four letters by e or f; for suppose that the substitution represented by s_α has this property, replacing, for instance, b by e, then would $s_1 t$ replace a by e, which is contrary to hypothesis. Hence e and f can only replace each other in the substitutions of G. Since this proof could clearly be used with respect to any number of letters we have arrived at the following important theorem:

Theorem 1. In every intransitive group the letters may be divided into such systems that the substitutions of the group will connect the letters of the systems transitively and no substitution will replace a letter of one system by one of another system.†

Since each of these systems must form a transitive group it follows that *every intransitive group can be formed by combining transitive groups, such that the sum of the letters in these groups is equal to the number of letters in the intransitive group.*

The main problem before us is, therefore, the development of such methods as are most helpful in combining the transitive groups. The two intransitive groups in the list of the groups of four letters are instances of the simplest methods of combination. One of these

$$1,\ ac,\ bd,\ ac.bd$$

is obtained by multiplying every substitution of one group (ac)* by every substitution of the other (bd). It is evident that by this method we can always construct an intransitive group from two or more groups each involving different letters, *e. g.* the intransitive group of five letters which may be obtained in this way from (abc) and (de) is

$$1,\ abc,\ acb,\ de,\ abc.de,\ acb.de$$

The other intransitive group in the given list

$$1,\ ab.cd$$

is obtained by the process called simple *isomorphism* or 1, 1 correspondence. The process consists in associating substitutions of the component transitive groups which have the same properties with respect to the groups. The given intransitive group is obtained from the groups

$$\begin{array}{cc} 1 & 1 \\ ab & cd \end{array}$$

It is at once evident that 1 may be associated with 1 and ab with cd, and we thus obtain the required group

$$\begin{array}{c} 1 \\ ab.cd \end{array}$$

†cf. Jordan: Traite des Substitutions, Art. 40: also Netto's Theory of Substitutions (Cole's edition), page 70.

*The parenthesis is used to indicate the group generated by the substitution enclosed: thus, (ac)= 1. ac; (abc)= 1, abc, acb; etc. In case the group consists of more than two substitutions the abbreviation for cyclical is commonly written after the parenthesis: thus, (abc)=(abc) *cyc.*

Similarly we may obtain an intransitive group of six letters from (abc) and (def), viz.

$$1,\ abc.def,\ acb.dfe$$

Instead of a 1, 1 correspondence we may have an a, b correspondence, a and b representing any positive integers. In this way we obtain the intransitive group

$$1,\ abc,\ acb,\ ab.de,\ ac.de,\ bc.de$$

from the two groups

$$1,\ abc,\ acb,\ ab,\ ac,\ bc$$

and $1,\ de$

by a 3, 1 correspondence.

We have now given the main methods employed in constructing intransitive groups. We proceed to find all the

Intransitive Groups of Five Letters.

All these groups are composed of a transitive group of three and another of two letters: for the only way of dividing five so as to get two or more letters in each system is to divide it into the parts three and two. The component groups are therefore,

$$\left.\begin{array}{l} 1,\ abc,\ acb \\ 1,\ abc,\ acb,\ ab,\ ac,\ bc \end{array}\right|\ 1,\ de$$

It is evident that (de) can be combined with (abc) only by multiplying the two groups together. We thus obtain the intransitive group given above as an illustrative example.

By combining (de) with the second group on the left we obtain, in a similar way,

$$\begin{array}{lllll} 1 & abc & abc.de & ab.de & de \\ & acb & acb.de & ac.de & ab \\ & & & bc.de & ac \\ & & & & bc \end{array}$$

In this case the combination may be effected in one more way. Since the first half of the substitutions in the second group on the left form a subgroup we may let 1 of (de) correspond to these and de to the remaining substitutions. We thus obtain the intransitive group of five letters which was given above as an illustrative example of an a, b correspondence.

From this we see that there are only three intransitive groups in five letters. In Professor Cayley's list these groups are denoted by

$$(abc)cyc.(de),\ (abc)all(de),\ \{ (abc)all(de) \}\text{pos.}$$

There are twenty-one intransitive groups involving six letters. The component groups may involve any of the following systems of letters:

2, 2, 2,
4, 2
3, 3

We expect to apply the given methods to the construction of all these groups. This, it is believed, will give sufficient exercise in the construction of this class of groups and we shall then proceed to the construction of the *transi-*

tive groups.

The reader who desires a thorough working knowledge of this subject could very profitably work over this field and compare his methods and results with those that we shall give.

THE GOLDEN SECTION.

By EMMA C. ACKERMANN, Instructor in Mathematics, Michigan State Normal School.

In the number for November 1892, of *Lehrproben und Lehrgange aus der Praxis der Gymnasien und Realschulen*, there appeared an article by Prof. Dr. O. Willman, entitled Der goldene Schnitt abs ein Thema des mathematischen Unterrichts. The article is interesting not alone to students of geometry, but to all who are at all concerned with the question of concentration, a question which is at present creating for itself an active interest among all educators. The article is a resume of a work on the golden section by F. C. Pfeifer, Der goldene Schnitt und dessen Erscheinungsformen in Mathematik, Natur, und Kunst, Augsburg, Huttler, 1885. The statements here presented are intended as a summary of the article.

It is very necessary that the connection between instruction in mathematics and in the remaining courses of study should be carefully considered because the subject of mathematics is an abstract one and according to its nature tends towards isolation.

To assist in bringing about this connection, there should be prepared mathematical problems and exercises which will show the application of mathematics to technics and to observations in nature on the one hand, and on the other furnish an insight into the history of mathematics, by means of which historical and classical instruction can be connected with the mathematical. A subject which meets these demands and is at the same time well adapted for purely mathematical instruction is the theme of the golden section, a theme which does not appear in a complete form in our modern text-books.

The simplest division of any magnitude, involving the fewest conditions is the division into two equal parts. Calling a line so divided, S, the parts p, we have $S=2p$, $p=\frac{S}{2}$, $\frac{S}{p}=2$, $\frac{p}{S}=\frac{1}{2}$, $\frac{p}{p}=1$. Contrasted to one case of division into two equal parts stands an infinite number of divisions into two unequal parts; and the ratio of the smaller (m) to the larger (M), $\frac{M}{m}$, or the ratio of one of the parts to the whole, $\frac{m}{S}$ or $\frac{m}{m+M}$ and $\frac{M}{S}$ or $\frac{M}{m+M}$ can be expressed by many different numbers. In one case only is there no need of figures to determine the ratio of the parts to the whole; and that is when

$\frac{m}{M}=\frac{M}{m+M}$. Such a division constitutes the golden section.

From the proportion $m:M=M:m+M$, we have $mM+m^2=M^2$ or $(M+m)(M-m)=mM$; or in the golden section, the sum of the parts multiplied by their difference equals their product. It also follows that the greater part is the geometric mean between the smaller part and the whole. Let $S=m+M$; then $S-m=\sqrt{Sm}$; or the difference between the whole and the smaller part is the geometric mean of those two parts. Also both parts form with the whole a continued proportion, $m:M=M:m+M$ or $m:M:m+M$, distinguished from all other proportions by the fact that the third quantity is at the same time the sum of the other two. From $\frac{M}{m}=\frac{M+m}{M}$, we have $\frac{M}{m}=1+\frac{m}{M}$ or $\frac{m}{M}=\frac{M}{m}-1$; that is, the quotient of the two parts is greater or less than its reciprocal by unity. Call the ratio $\frac{M}{m}$, e, then we have the equation $e=1+\frac{1}{e}$, solving $e=\frac{\sqrt{5}+1}{2}=1.61803+$.

Of the three elements of the golden section, each two can be expressed by the third. The simplest is that of m and S by means of M; $m=\frac{\sqrt{5}-1}{2}M$; $S=\frac{\sqrt{5}+1}{2}M$. Also $m=\frac{3-\sqrt{5}}{2}S$ and $M=\frac{\sqrt{5}-1}{2}S$, that is, the major part becomes the minor, when the whole is considered the major. If m is the base, $M=\frac{\sqrt{5}+1}{2}m$, $S=\frac{\sqrt{5}+3}{2}m$.

For the construction and consideration of the golden section, the number 5, which appears in the value of e is very suggestive. $5=1+4$, and therefore can be expressed by the Pythagorean theorem; 5 is the area of the square on the hypotenuse, if $1+4$ are the squares on the other two sides. The sides themselves are 1 and 2, and the hypotenuse, $\sqrt{5}$. Half of $\sqrt{5}$ increased by $\frac{1}{2}$ of unity will then express the value of e, and diminished by $\frac{1}{2}$ of unity, its reciprocal. A line AB is divided into medial section therefore, if in the right triangle ABC, with right angle ABC, we make $BC=\frac{1}{2}AB$, draw AC, lay off $CD=BC$, and then lay off the remainder AD or AB as AE; then AB is divided into medial section.

The number 5 may be used in another way to illustrate the ratio of m to M. An isosceles triangle with angles $\frac{\pi}{5}$, $\frac{2\pi}{5}$, $\frac{2\pi}{5}$ is constructed. By bisecting one of the equal angles, we have a triangle similar to the first. Let the triangles be ABC and ABD respectively, angle B being $\frac{\pi}{5}$; then CB is divided in medial section, from principles of similar triangles. The triangle ABC is

middle part of a pentagon which is completed by placing two triangles congruent to triangle ABD on AB and BC. AD produced will then pass through an angle of the pentagon and CD becomes the smaller part and BD the larger part of a diagonal. Therefore in a regular polygon of five sides the smaller part of a diagonal cut off by a second diagonal forms with the side of the pentagon and the diagonal, the proportion of the golden section. The triangle is also an element of the regular decagon and will produce it if repeated ten times.

In the division of magnitudes into two equal parts, the whole may be considered as one of the parts repeated; so in the golden section, each one of the parts may be considered as the starting-point and the next as a repetition of it augmented or diminished. If we proceed from the minor part, the major is a repetition of the minor increased, and the sum of the two bears the same relation to the major, the ratio in each case being e. So if we proceed from the whole to the major, and from that to the minor. With this view of the case, there is no necessity for stopping with three elements, since this augmenting or diminishing repetition can evidently be carried on indefinitely. In this way the geometric proportion of the golden section becomes a geometric progression which from analogy is called the golden progression, the ratio being e or $\frac{1}{e}$. The golden progression differs from the other geometric series in this that each of its members is also the sum of the two preceding.

If its first term is a, then this progression has the form a, ae, ae^2, ae^3, $ae^4 \ldots ae^n$. or a, ae, $a+ae$, $a+2ae$, $2a+3ae$, $3a+5ae \ldots$

Then $a^2=a(1+e)$; $ae^3=a+2ae$; $ae^4=a(2+3e)$, etc. If $a=1$, then since $e=\frac{\sqrt{5}+1}{2}=1.61803$, the series is: 1, 1.61803+, 2.16803+, 4.23607+, 6.185410+, 11.09017+, 17.94427+, 29.03444+, etc. If we should tentatively place e also equal to 1, that is, 0.61803 too small, then as a geometric progression simply, the series remain stationary. But using the other property of the golden progression, the series becomes 1, 1, 2, 3, 5, 8, 13, etc. The quotient of any two successive members is alternately smaller and larger than e, but as the series advances, the quotient approaches nearer to e, as at $\frac{233}{144}=1.618$. This shows that a series beginning with the smallest natural numbers and advancing according to the second condition above, forms in its continuation an approximation to the golden progression.

The golden progression can be represented graphically thus: A pentagon is drawn whose sides are F and diagonals D; the intersections of the diagonals determine a pentagon whose sides are f and whose angles are at a distance a from the vertices of the original polygon. Then f, a, F, D form a golden series. By making the original polygon the enclosed polygon of a larger pentagon, whose sides and diagonals are F' and D' respectively, we can continue the progression as an ascending series; or, by drawing polygons within the pentagon, as a decending series.

Another interesting figure might be given here. The line AB is divided in medial section at C, AC being the major part; perpendiculars BD,

CF, of length AC are drawn at B and C, and are divided at E and G respectively into medial section, BE and GC being the major parts. Then AB, AC $(=BD=CF)$, $BC(=BE=CG)$ and $DE(=FG)$ form a golden series. Connect F and A, G and A, E and A. Let AC be taken as a radius or unity; AC then represents the side of a hexagon; AF the side of a square; AG, of a regular pentagon, EB or CG, the side of a decagon; and AE, the side of a regular triangle, all of which is evident from the right angled triangle.

Another figure may be obtained by making m, M, $m+M$ the radii of concentric circles. Their areas are then πm^2, πM^2, $\pi(m+M)^2$, the area of the inner ring, r, is $\pi(M^2-m^2)$ and of the outer ring, R, $\pi[(M+m)^2-M^2]$. If $m=1$, then $M=e$, $m+M=e^2$, $(M+m)^2=e^4$, $M^2-m^2=e^2-1$, and $(M+m)^2-M^2=e^4-e^2=e^2(e^2-1)=e^3$. Then the following series arises:

$\pi=$ area of inner circle, f;
$e\pi=$ " " " ring, r;
$e^2\pi=$ " " middle circle, F;
$e^3\pi=$ " " outer ring, R;
$e^4\pi=$ " " " , circle, F'.

This series can be extended both as increasing and decreasing; the members with even exponents as e^6, e^8, e^{-2}, e^{-4} correspond to circles; those with odd exponents to rings.

Countless illustrations of the proportions of the golden section are found in nature and the works of man. The golden section follows closely upon bisection (the basis of symmetry) everywhere, and the forms which are based upon the proportions of the golden section though not so evident are more widely distributed than would appear at first thought. Whenever, in the products of art or manufacture, there is no equal division, (symmetry), the artist or workman unconsciously employs the proportions of the golden section. Irregular inequality and capricious division is disagreeable to both eye and hand; and the porportion of the golden section seem to be the only acceptable ones. Accordingly, the form of writing-paper, books, a page of the MATHEMATICAL MONTHLY, furniture, especially tables and chairs, doors, windows, dimensions of pictures, foundations and often the facades of buildings, all reveal these proportions.

This is true of not only modern art and technics, but also of the ancient. We find the same proportions in the pyramids of Cheops, in the temples at Karnak and at Ombos, in the Grecian temples, and many cathedrals.

In verses of poetry and in music, the same relation is found, and most abundantly in nature. In leaves, plants, lower animals, and man, these proportions have been verified.

The subject of the golden section is not discussed by Euclid; he had a knowledge of it and mentions it in his works, though not under this name. The name, though it has an ancient ring, is not found in ancient literature. Aristotle does not mention the subject, but it is claimed that in his philosophical reasoning, there rules the principle of the golden section; *i. e.*: the relation of the whole to the part and the parts to each other. His ideas were not carried

out by the ancient philosophers but they were the source of much of the speculation in mediaeval times, when mathematical and philosophical thought were closely allied. One writer, John Campanus of Novava, thought that the principle of the golden section descended from the gods. Keplertcompared it to a precious stone, and called it *proportio divina*, but not *proportio* or *sectio aurea*. The latter name has originated since his time.

THE RECTIFICATION OF THE CASSINIAN OVAL BY MEANS OF ELLIPTIC FUNCTIONS.

By F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

(*Continued from the July-August Number.*)

The central-polar equation of the Cassinian Oval may be written

$$r^4-(2c^2\cos 2\theta)r^2=m^4-c^4\ldots\ldots(1).$$

$$\therefore \cos 2\theta=\frac{r^4-(m^4-c^4)}{2c^2r^2},\text{ and }\sin 2\theta=\sqrt{\left(\frac{4c^4r^4-[r^4-(m^4-c^4)]}{4c^4r^4}\right)}.$$

$$\therefore \mathsf{P}=8m^2\int_b^a\frac{r^2dr}{\surd\{4c^4r^4-[r^4-(m^4-c^4)]^2\}}$$

$$=8m^2\int_b^a\frac{r^2dr}{\surd\{[(m^2+c^2)^2-r^4]\times[r^4-(m^2-c^2)^2]\}}\ldots\ldots(2).$$

Reducing (2) under the supposition that

$$r^4=(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi,$$

$$\mathsf{P}=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{r}=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^2+c^2)^2\cos^2\phi+(m^2-c^2)^2\sin^2\phi]^{\frac{1}{4}}}$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^2+c^2)^2-4m^2c^2\sin^2\phi]^{\frac{1}{4}}}\ldots\ldots(j),$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^4+c^4)+2m^2c^2(1-2\sin^2\phi)]^{\frac{1}{4}}}$$

$$=4m^2\int_o^{\frac{1}{2}\pi}\frac{d\phi}{[(m^4+c^4)+2m^2c^2\cos 2\phi]^{\frac{1}{4}}}\ldots\ldots(3).$$

Let $2\phi=\psi$, and make $2m^2c^2/(m^4+c^4)=C$; then, after obvious transformations, (3) gives

$$\mathsf{P}=\frac{2m^2}{[m^4+c^4]^{\frac{1}{4}}}\int_0^\pi\frac{d\psi}{[1+C\cos\psi]^{\frac{1}{2}}}\ldots.(4).$$

After expanding (4) into a series of not less than two dozen terms, and observing that the negative terms of the series will *vanish* on taking the integral limits, we obtain a series expressing the perimeter of the Cassinian Oval. Since $m^2=5$ and $c^2=4$; that is, since the semi-axes of the Cassinian Oval in consideration are 3 and 1 linear units, we have $C=\frac{40}{41}$. After a rather laborious calculation, we find $\mathsf{P}=14.9831+$ linear units. On page 223 of the July-August MONTHLY, $C=\frac{80}{81}$; and four terms of that resulting series give a perimeter ($\mathsf{P}=12.7329+$ linear units) too small by $2\frac{1}{4}$ linear units. Since the *moduli* of these functions are almost unity, the resulting series will not converge rapidly; and with this same trouble, it must be remembered, *M.* Legendre also had to contend. Possibly some of the talented readers of the MONTHLY will succeed in expanding (j), or (4), into a *rapidly-converging* series.

After performing certain rather elaborate transformations of *premises approximative in derivation*, we deduce the following two remarkable formulae for the perimeter of the Cassinian Oval:

II. Transforming the Cartesian equation of the Cassinian Oval by the formulae, $x=r\cos\theta$ and $y=r\sin\theta$, we have

$$r^2=\sqrt{(m^4-c^4\sin^2 2\theta)}+c^2\cos 2\theta\ldots.(1).$$

$$\therefore\ rdr=\frac{-c^2[\sqrt{(m^4-c^4\sin^2 2\theta)}+c^2\cos 2\theta]\sin 2\theta d\theta}{\sqrt{(m^4-c^4\sin^2 2\theta)}}\ldots.(\alpha),$$

$$\text{and}\left(\frac{dr}{d\theta}\right)^2=\frac{c^4[\sqrt{(m^4-c^4\sin^2 2\theta)}+c^2\cos 2\theta]\sin^2 2\theta}{m^4-c^4\sin^2 2\theta}\ldots.(\beta).$$

$$\therefore\ \mathsf{P}=4m^2\int_0^{\frac{1}{2}\pi}\sqrt{\left(\frac{\sqrt{(m^4-c^4\sin^2 2\theta)}+c^2\cos 2\theta}{m^4-c^4\sin^2 2\theta}\right)}d\theta\ldots.(2).$$

Put $(c^2/m^2)^2=C^2$; then (2) can easily be transformed into

$$\mathsf{P}=4m\int_0^{\frac{1}{2}\pi}\sqrt{\left(\frac{\sqrt{(1-C^2\sin^2 2\theta)}+C\sqrt{(1-\sin^2 2\theta)}}{1-C^2\sin^2 2\theta}\right)}d\theta$$

$$=4m\sqrt{(1+C)}\int_0^{\frac{1}{2}\pi}\sqrt{\left[1+\left(\frac{2(1+C^2)-(2+C)}{2}\right)\sin^2 2\theta\right]}d\theta$$

$$=4m\sqrt{\left(\frac{(1+C)[2(1+C^2)-C]}{2}\right)}\int_0^{\frac{1}{2}\pi}\sqrt{\left[1-\left(1-\frac{2}{2(1+C^2)-C}\right)\sin^2\phi\right]}d\phi,$$

$$=\tfrac{2}{3}\pi m\sqrt{\left(\frac{(1+C)[2(1+C^2)-C]}{2}\right)}\left[1-\Sigma\left(\frac{1.3.5\ldots.(2n-1)}{2.4.6\ldots.2n}\right)^2\frac{M^{2n}}{2n-1}\right];$$

of which elliptic function M is the *modulus*, and n may have all consecutive integral values from *unity* to *infinity*. When $m^2=5$ and $c^2=4$, $\mathsf{P}=14.9652$.

III. From the Cartesian equation of the Cassinian Oval, we deduce

$$y^2=\sqrt{(m^4+4c^2x^2)}-(c^2+x^2)\ldots.(3),$$

an equation which gives all the *real* points of the Oval in consideration.

$$\therefore\left(\frac{dy}{dx}\right)^2=\frac{[4c^4-4c^2\sqrt{(m^4+4c^2x^2)}+m^4+4c^2x^2]x^2}{(m^4+4c^2x^2)[\sqrt{(m^4+4c^2x^2)}-(c^2+x^2)]}\ldots.(\gamma).$$

Representing the semi-axis major of the Cassinian Oval by $a_1=\sqrt{(m^2+c^2)}$, we have

$$\mathrm{P}=4\left(\frac{2c^2-m^2}{m^2\sqrt{(m^2-c^2)}}\right)\int_0^{a_1}\sqrt{\left(\frac{m^4(m^2-c^2)}{(2c^2-m^2)^2}+x^2\right)}dx$$

$$=2\left[\left(\frac{\sqrt{[m^4(m^2-c^2)+(m^2+c^2)(2c^2-m^2)^2]}}{m^2}\right)\sqrt{\left(\frac{m^2+c^2}{m^2-c^2}\right)}\right.$$

$$+\frac{m^2\sqrt{(m^2-c^2)}}{2c^2-m^2}$$

$$\left.\log\left(\frac{(2c^2-m^2)\sqrt{(m^2+c^2)}+\sqrt{[m^4(m^2-c^2)+(m^2+c^2)(2c^2-m^2)^2]}}{m^2\sqrt{(m^2-c^2)}}\right)\right],$$

$$=\frac{16}{5}\left[\frac{3\sqrt{(106)}}{5}+\frac{5}{3}\log\left(\frac{9+\sqrt{(106)}}{5}\right)\right]=14.9833,$$

when $m^2=5$ and $c^2=4$.

[To be continued.]

POSTULATE II. OF EUCLID'S ELEMENTS.

By Professor JOHN N. LYLE, Ph. D., Westminister College, Fulton, Missouri.

"Let it be granted that a terminated straight line may be produced to any length in a straight line."

Euclid lays down the statement just quoted as his second postulate regulative of geometrical constructions. Wherever in unbounded space any point may be located to which a straight line has been extended, Euclid assumes that the straight line may be lengthened out beyond that point.

Riemann assumes that every straight line is finite in length, and if extended will ultimately return to the starting point.

If a straight line that is produced from a given point eventually returns to the same point, Euclid's postulate 2 is false.

On the other hand, if the second postulate of Euclid is true, the Rie-

mannian hypothesis that contradicts it must be false. This follows inevitably by the logical law of *Excluded Middle*, according to which if one of two propositions that mutually contradict each other is true, the other must be false.

According to the Euclidian view the longer a straight line is the further apart are its ends.

According to the Riemannian view a straight line may be lengthened until its ends approach and ultimately meet.

The hypothesis of Riemann and the 2nd postulate of Euclid contradict each other. Hence, both cannot be true. To accept both is to discredit logical law. To say that we do not know which is true is to confess that we are not in possession of geometrical Science.

According to the laws of logical deduction, if Euclid's postulate 2 is false, the geometrical System derived from it is not true.

On the other hand, if the assumption that contradicts Euclid's postulate 2 is false, the system logically deduced from it is not true. Sound geometrical propositions are not obtained by logical deduction from false data.

According to the Riemannian hypothesis the angle sum of a rectilineal triangle is greater than two right angles. But Lobatschewsky proves in his theorem 19 that the angle sum of a rectilineal triangle cannot be greater than two right angles. The hypotheses of Lobatschewsky and Riemann, therefore, are seen to clash with each other as well as with the axioms, postulates and theorems of Euclid's Elements.

The chords of arcs of circles are not identical with the arcs subtended by them. Hence *rectilineal* triangles should not be treated as identical with *spherical* triangles. This statement holds whatever the length of the radius of the sphere may be. The radius of every sphere has *two* ends, one at the centre and the other at the surface. But every straight line with *two* ends is *finite*. We are now face to face with Postulate III. of Euclid's Elements.

SUBSTITUTION GROUPS.

THE CONSTRUCTION OF INTRANSITIVE GROUPS CONTINUED.

Before seeking all of the possible intransitive groups of degree* six it seems well to call attention to several facts which may be employed to advantage in this work. To illustrate we shall employ a group which was given before, viz.

* The degree of a group is equal to the number of letters it involves. Thus $(abcd)$ pos is of the fourth degree.

$$
\begin{array}{llll}
 & & abc & acb \\
(abcd)\text{pos.}\dagger = 1 & ab.cd & bdc & bcd \\
 & ac.bd & adb & abd \\
 & ad.bc & acd & adc.
\end{array}
$$

If t and s represent any two substitutions then is

$$t^{-1}st$$

(where t^{-1} represents the substitution which reverses the operation indicated by t) called the *conjugate of s with respect to t.* t and t^{-1} are said to be the *inverse* of each other, since $tt^{-1} = t^{-1}t = 1$.

We proceed to find the conjugates of $ab.cd$ with respect to the other substitutions of $(abcd)$ pos. We obtain the following results:

$$
\begin{array}{lll}
ac.bd & ab.cd & ac.bd = ab.cd \\
abc & ab.cd & acb = ac.bd \\
acb & ab.cd & abc = ad.bc\ddagger
\end{array}
$$

It can readily be veritid that all the substitutions in one of the above columns transform $ab.cd$ into the same substitution.

Definition. If all the substitutions of a group transform all the substitutions of a subgroup into substitutions of the subgroup, the subgroup is called a self conjugate subgroup of the given group.

$$1, \quad ab.cd, \quad ac.bd, \quad ad.bc$$

constitute a self conjugate subgroup of $(abcd)$ pos, while the subgroups

$$1, \quad ab.cd \quad \text{and} \quad 1, \quad abc, \quad acb$$

are not self conjugate. If we exclude identity and the entire group from the subgroups, it can easily be verified that only one of the eight subgroups of the given group is self conjugate.

Since all the intransitive groups of a given degree n can be obtained by combining transitive groups such that the sum of their degrees is equal to n it follows that all the intransitive groups of degree six can be found by combining

(1) a transitive group of degree three with a transitive group of degree three,

(2) a transitive group of dregree two with a transitive group of degree four, and

(3) a transitive group of degree two with two transitive groups of degree two.*

We proceed to find the intransitive groups for each of these divisions separately. We shall thus not only find all the groups but also each group only once since it is evidently impossible for one group to belong to two of these divisions.

[To be continued.]

†By $(abc \ldots l)$ all we mean all the substitutions that can be formed with the letters $a, b, c, \ldots, l$ and by $(abc \ldots l)$ pos we mean the subgroup of the preceding group which involves only its positive substitutions; i. e. all its substitutions which indicate an even number of interchanges of two letters. In place of (abc) pos it is customary to write (abc) cyc or merely (abc).

‡The conjugate may be obtained by replacing each letter by the letter which follows it in the substitution with respect to which it is conjugate. For let $s = a_1 a_2 a_3 \ldots$ and let $b_1 b_2 b_3 \ldots$ be the letters in order which in t succeed the given letters of s. Then $t^{-1}st$ replaces b_1 by a_1, a_1 by a_2, and a_2 by b_2; i. e. it replaces b_1 by b_2 and similarly it replaces b_2 by b_3. If a_α is not explicitedly found in t we have to observe that $b_\alpha = a_\alpha$ in using this method.

*Every group is transitive whose degree (n) satisfies the inequality $n < 4$.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

50. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

If A can walk to the city and ride back, he will require $m=5\frac{1}{4}$ hours; but if he walk both ways, he will require $n=7$ hours. How many hours will he require to ride both ways?

I. Solution by A. L. FOOTE, Middleburg, Connecticut.

Taking it for granted that he can walk or ride either way with equal facility, we find that he could walk to the city in $\frac{n}{2}=3\frac{1}{2}$ hours and can ride back in $m-\frac{n}{2}=5\frac{1}{4}-3\frac{1}{2}=1\frac{3}{4}$ and he can also ride to the city in $1\frac{3}{4}$ hours, so he will take $1\frac{3}{4}\times2=3\frac{1}{2}$ hours. On any other supposition the problem is indeterminate.

II. Solution by H. C. WILKS, Murraysville, West Virginia, and J. F. W. SCHEFFER, A. M., Hagerstown, Maryland

A can walk up and walk back in $n=7$ hours. He can walk up and ride back in $m=5\frac{1}{4}$ hours.

$\therefore$ times of walking back and riding back differ by $n-m=1\frac{3}{4}$ hours.

Also times of walking round trip and riding round trip differ by $2(n-m)=3\frac{1}{2}$ hours. But he *walks* round trip in $n=7$ hours.

Hence he *rides* round trip in $n-2(n-m)=7-3\frac{1}{2}$ or in $2m-n=3\frac{1}{2}$ hours.

III. Solution by Professor P. S. BERG, Larimore, North Dakota.

To walk one way he will require $\frac{n}{2}$ hours. Hence to ride one way he will require $\left(m-\frac{n}{2}\right)$ hours, and to ride both ways he will require

$$2\left(m-\frac{n}{2}\right)=3\tfrac{1}{2}\text{ hours.}$$

IV. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Vice President in Texarkana College, Texarkana, Arkansas.

$\frac{n}{2}$ = number of hours to walk one way, and

$m-\frac{n}{2}=\frac{2m-n}{2}$ = number of hours to ride one way.

$\therefore 2\left(\frac{2m-n}{2}\right)=2m-n$ = number of hours to ride both ways.

But $m=5\frac{1}{4}$, $n=7$.

$\therefore 3\frac{1}{2}$ hours = required time.

This problem was also solved by *Professor COOPER D. SCHMITT*, ——, and the *PROPOSER*.

51. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A banker, in discounting a note due in $m=4$ months at $r=3\%$ per annum charges $C=\$12\frac{1}{4}$ more than the true discount. What is the face of the note discounted?

I. Solution by G. B. M. ZEER, A. M., Ph. D., Vice President and Professor of Mathematics in the Texarkana College, Arkansas.

$$m \text{ months } +3 \text{ days grace} = \frac{10m+1}{10} \text{ months.}$$

$$\frac{(10m+1)r}{12000} = \text{bank discount on \$1 at } r\% \text{ for } \frac{10m+1}{10} \text{ months.}$$

$$\frac{1200+mr}{1200} = \text{amount of \$1 at } r\% \text{ for } m \text{ months.}$$

$$\frac{1200}{1200+mr} = \text{proceeds of \$1 at } r\% \text{ for } m \text{ months.}$$

$$1-\frac{1200}{1200+mr} = \frac{mr}{1200+mr} = \text{true discount.}$$

$$\frac{(10m+1)r}{12000} - \frac{mr}{1200+mr} = \frac{(1200+10m^2r+mr)r}{12000(1200+mr)} = \text{difference.}$$

$$\frac{12000(1200+mr)C}{(1200+10m^2r+mr)r} = \text{face of note.}$$

Substituting $m=4$, $r=3$, $C=12\frac{1}{4}$, we get face of note=\$35099.29.

II. Solution by the PROPOSER.

Represent the face of the note by X, and the number of days of grace by g; then the *bank discount* is $(30m+g)rX/36000$, and the *true discount* is $mrX/(1200+mr)$. According to the problem,

$$\left[\frac{30m+g}{36000} - \frac{m}{1200+mr}\right]rX = C \ldots . (1),$$

$$\therefore X = \frac{36000(1200+mr)}{[(30m+g)mr+1200g]r} \text{ of } \$C, =\$35099.2908,$$ which is the face of the note required.

NOTE—Make $g=0$; then $X=123725.00$, which is the result of Miscellaneous Problem, No. 844, *Lock and Scott's Arithmetic for Schools*, page 281.

III. Solution by COOPER D. SCHMITT. Professor of Mathematics in University of Tennesee, Knoxville, Tennesee.

Interest for 4 mon. 3 da. at 3% on any principal is $\frac{123}{12000}$ of the principal and bank discount is same as simple interest.

In true discount, the three days of grace are not counted.

The true discount on any principal for $\frac{1}{3}$ of a year at 3 per cent. is the same as for one year at 1%, which is $\frac{1}{101}$ of the principal.

Hence, $(\frac{123}{12000}-\frac{1}{101})$ of the principal=\$12$\frac{1}{4}$, or $\frac{423}{1212000}$ of principal=\$12$\frac{1}{4}$, and the principal=$\frac{1212000}{423}$ of 12$\frac{1}{4}$=\$35099.29.

This problem was solved with different results by *P. S. BERG. J. F. W. SCHEFFER. A. L. FOOTE* and——.

PROBLEMS.

54. **Proposed by D. P. WAGONER, A. B., Principal of the School of Languages, Westerville, Ohio.**

A man bought a farm for $6,000 and agreed to pay for it in four equal annual installments at 6% annual interest compounded every instant. Required his annual payment. *B. F. Burleson.*

55. **Proposed by J. C. CORBIN, Pine Bluff, Arkansas.**

How long will it take to count a million, in the following manner: the counter is to pronounce each syllable in the names of the successive numbers at the rate of one per second?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

46. **Proposed by WILLIAM HOOVER, A. M., Ph. D, Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

Find θ from $\cos\theta+\cos 3\theta+\cos 5\theta=0\ldots.(1)$.

Solutions by JOE A. JOHNSON, Student of the Sophomore Class, University of Mississippi; E. L. SHERWOOD, A. M., Professor of Mathematics, Mississippi Normal College, Houston, Mississippi; J. B. FAUGHT, A. B, Indiana University, Bloomington, Indiana; J. C. CORBIN, Pine Bluff, Arkansas; G. I. HOPKINS, Department of Mathematics in High School, Manchester, N. H.; OTTO CLAYTON, M. B., Maxwell, Indiana; O. W. ANTHONY, M. S, Missouri Military Academy, Mexico, Missouri; and A. H. BELL, Hillsboro, Illinois.

I. Equation (1) $=\cos\theta+(4\cos^3\theta-3\cos\theta)+(16\cos^5\theta-20\cos^3\theta+5\cos\theta)=16\cos^5\theta-16\cos^3\theta+3\cos\theta=\cos\theta\,(16\cos^4\theta-16\cos^2\theta+3)=0\ldots.(2)$. Whence $\cos\theta=0$. $\therefore\ \theta=\frac{\pi}{2}$. From $16\cos^4\theta-16\cos^2\theta+3=0$, we get $\cos^2\theta=\frac{3}{4}$, or $\frac{1}{4}$. $\therefore\ \cos\theta=\pm\frac{1}{2}\sqrt{3},\ \pm\frac{1}{2}$. $\therefore\ \theta=\frac{\pi}{6},\ \frac{5}{6}\pi$, or $\frac{\pi}{3}$ and $\frac{2}{3}\pi$, which are all the values $<\pi$. [*J. A. Johnson.*]

II. Factoring (2), $(4\cos^2\theta-3)(4\cos^2\theta-1)\cos\theta=0$. Whence $\cos\theta=\pm\frac{1}{2}\sqrt{3},\ \pm\frac{1}{2},\ 0$. $\therefore\ \theta=\frac{\pi}{6},\ \frac{\pi}{6}\pi,\ \frac{5}{6}\pi,\ \frac{7}{6}\pi;\ \frac{\pi}{3},\ \frac{5}{3}\pi,\ \frac{2}{3}\pi,\ \frac{4}{3}\pi;\ \frac{1}{2}\pi,\ \frac{3}{2}\pi$. [*E. L. Sherwood.*]

III. From equation (2), we find $\cos\theta=0$; $\pm\frac{1}{2}\sqrt{3}$, or $\pm\frac{1}{2}$. $\therefore\ \theta=\frac{\pi}{2}$,

$\frac{\pi}{6}$, $\pi\pm\frac{\pi}{6}$, $\pm\frac{\pi}{3}$, $\pi\pm\frac{\pi}{3}$; or in *general*, $n\pi\pm\frac{\pi}{2}$, $2n\pi\pm\frac{\pi}{6}$, $(2n+1)\pi\pm\frac{\pi}{6}$, $2n\pi\pm\frac{\pi}{3}$, $(2n+1)\pi\pm\frac{\pi}{3}$. [*J. B. Faught.*]

IV. From equation (2), $\cos\theta=0$. $\therefore$ $\theta=90^\circ, 270^\circ, \ldots$. If $x=\cos^2\theta$, the second factor becomes $x^2-x=-\frac{3}{16}$, and $x_1=\frac{3}{4}$, $x_2=\frac{1}{4}$. $\therefore$ $\cos\theta=\frac{1}{2}\sqrt{3}$ and $\cos\theta=\frac{1}{2}$. $\therefore$ $\theta=30^\circ, 330^\circ \ldots$, and $\theta=60^\circ, 300^\circ \ldots$, for *particular* values. [*J. C. Corbin.*]

V. Contracting the last two terms of (1) into a *product of cosines*, by the formula $\cos x+\cos y=2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$, we have $\cos\theta+2\cos4\theta\cos\theta=0$, or $\cos\theta(1+2\cos4\theta)=0$, $\therefore$ $\cos\theta=0$, and $\theta=\frac{\pi}{2}$ or 90°, or in general $\frac{1}{2}(2n\pm1)$; also, $1+2\cos4\theta=0$, $\therefore$ $\cos4\theta=-\frac{1}{2}$, and $\theta=30^\circ, 60^\circ$, or in general $\frac{1}{6}(3n\pm1)$. [*G. I. Hopkins, Otto Clayton.*]

VI. Equation $(1)=\dfrac{\cos3\theta\sin3\theta}{\sin\theta}=0$. $\therefore$ $\cos3\theta=0\ldots(a)$, or, $\sin3\theta=0\ldots(b)$. From (a), $\theta=\frac{1}{6}(2n+1)\pi$; from (b), $\theta=\frac{1}{3}n\pi$. [*O. W. Anthony.*]

VII. $\cos3\theta=\cos\theta\cos2\theta-\sin\theta\sin2\theta$; $\cos5=\cos\theta\cos4\theta-\sin\theta\sin4\theta$. $\sin\theta=\sqrt{1-\cos^2\theta}$, $\sin2\theta=2\cos\theta\sqrt{1-\cos^2\theta}$; $\sin4\theta=4\cos\theta(2\cos^2\theta-1)\sqrt{1-\cos^2\theta}$, $\cos2\theta=2\theta=2\cos^2\theta-1$. Substituting, etc., the given equation becomes $\cos4\theta-\cos^2\theta=-\frac{3}{16}$; whence $\cos\theta=\frac{1}{2}$ or $\frac{1}{2}\sqrt{3}$, giving $\theta=60^\circ$ or 30 for *particular* values. [*A. H. Bell.*]

NOTE.—The particular values given for this problem in Bowser's *Treatise on Trigonometry*, page 128, are $\frac{\pi}{2}$, $\frac{2}{3}\pi$.—*Editor.*

Also solved by *P. S. BERG, A. L FOOTE, F. P. MATZ, P. H. PHILBRICK, J. F. W. SCHEFFER C. D. SCHMITT, W. I. TAYLOR*, and *G. B M. ZEER.*

47. Proposed by LEONARD E. DICKSON, A. M., Fellow in Mathematics, University of Chicago, Chicago, Illinois.

Prove that $(-1)(-1)=+1$.

Solutions by the PROPOSER; G. B. M. ZEER, A. M, Ph. D., Professor of Mathematics in Inter State College, Texarkana, Texas; H. W. DRAUGHON, Olio, Mississippi; P. H. PHILBRICK, M. S., Chief Engineer for Kansas City, Watkins & Gulf Railway Co., Pineville, Louisiana; J H. GROVE, Professor of Mathematics in Howard Payne College, Brownwood, Texas; P. S. BERG, Apple Creek, Ohio; W. I. TAYLOR, Baldwin University, Berea, Ohio; Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland; A. L. FOOTE, C. E, Middlebury, Connecticut; and F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

I. *Assuming* the distributive law to hold, $(-1)\{(+1)+(-1)\}$, or 0,

$=(-1)(+1)+(-1)(-1)$. Assuming the commutative law, $(-1)(+1)=(+1)(-1)=-1$. $\therefore -1+(-1)(-1)=0$, or $(-1)(-1)=+1$.

This proof was suggested by a longer one due to Professor D. A. Hull of Upper Canada College [*L. E. Dickson.*]

II. $(-1)(-1)$ means that -1 is to be taken subtractively one time. $\therefore 0-(-1)=+1$. $\therefore (-1)(-1)=+1$. [*G. B. M. Zerr.*]

III. $-1\times a=-a$. $-1\times(a-1)=-(a-1)=-a+1$. $\therefore -1\times[(a-1)-a]=-a+1-(-a)=-a+1+a=1$ [*P. H. Philbrick.*]

IV. $(-1)(-1)=(-1)(+1)-(-1)(+2)=-1-(-2)=-1+2=+1$.
[*H. W. Draughon.*]

V. Definition: $-n$ is the number which added to $n=0$. We know that $(-1)\times 1=-1$; suppose $(-1)\times(-1)=x$. Adding we get, $(-1)(1-1)=x-1$. But $(-1)(1-1)=0$. $\therefore x-1=0$. $\therefore x=+1$. $\therefore (-1)(-1)=+1$.
[*J. H. Grove.*]

VI. To multiply one number by another we do to the first what is done to unity to produce the second. [See Smith's *Algebras*, Van Velzer and Slichter's *Univ. Alg.*] $\therefore (-5)(-3)=(-5)(-1-1-1)=-(-5)-(-5)-(-5)=+5+5+5=15$. Similarly, $(-1)(-1)=(-1)(-1)=-(-1)=+1$.
[*P. S. Berg*, *F. P. Matz.*]

VII. According to Wood's *Elements of Algebra, 17th edition*, we have $(-5)(-3)=+15$. Here -3 is to be subtracted 5 times; that is, -15 is to be *subtracted*. Now, subtracting -15 is the same as adding $+15$. Therefore, we have to *add* $+15$. Similarly, $(-1)(-1)=+1$.
[*W. L. Taylor*, *F. P. Matz.*]

VIII. The case$(-a)(-b)=+ab$ is purely conventional and consequently an assumption, which, however, does not deprive the result of its great importance to algebraic operations. [*J. F. W. Scheffer.*]

IX. For illustration, $(a-b)\times(c-b)=(c-b)a-(c-b)b$, but $(c-b)a=ac-ab$ and $(c-b)b=cb-b.b$, and we have $(a-b)(c-b)=ac-ab-(cb-bb)$. Now as we are to take cb less $b.b$ from $ac-ab$, we first take cb and we have $ac-ab-cb$; which is too much by $b.b$; we therefore add $b.b$ and get $ac-ab-cb+b.b$, but $b.b$ is found from $(-b)(-b)=+b.b$. Take $b=1$, then $(-1)(-1)=+1$.
[*A. L. Foote.*]

X. Revolve the vector $(+a)$ about its origin A, through an angle of 180°, and it will become the vector $(-a)$, or will be *multiplied* by (-1). Making a equal to unity, then revolving the vector (-1) about its origin A,

through an angle of 180°, and it will become the vector $(+1)$, or will be multiplied by (-1); that is, $(-1)(-1)=+1$. [*F. P. Matz.*]

PROBLEMS.

56. **Proposed by CHAS. E. MYERS, Canton, Ohio, and Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.**

(a). How much can be paid for a bond, bearing 5% interest and having ten years to run, so as to realize 3% on the investment? [*C. E. Myers*]; (b). At what price must the government sell 5% $100 bonds to run ten years, interest payable annually, to make them the same to the buyer as 3% bonds at par, to run ten years, interest payable anually, provided the buyer can invest all interest received at 4% interest payable anually? [*J. H. D.*]

57. **Proposed by J. C. CORBIN, Pine Bluff, Arkansas.**

Find the quotient of

$$\begin{vmatrix} (s-a_1)^2 & a_1^2 & a_1^2 \cdots\cdots & a_1^2 \\ a_2^2 & (s-a_2)^2 & a_2^2 \cdots\cdots & a_2^2 \\ a_3^2 & a_3^2 & (s-a_3)^2 \cdots\cdots & a_3^2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n^2 & a_n^2 & a_n^2 \cdots\cdots & s-a_n^2 \end{vmatrix} \div \begin{vmatrix} s-a_1 & a_1 & a_1 \cdots\cdots & a_1 \\ a_2 & s-a_2 & a_2 \cdots\cdots & a_2 \\ a_3 & a_3 & s-a_3 \cdots\cdots & a_3 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & a_n & a_n \cdots\cdots & s-a_n \end{vmatrix}$$

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

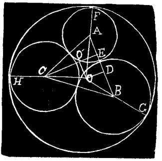
45. **Proposed by B. F. BURLESON, Oneida Castle, New York.**

Determine the radius of a circle circumscribing three tangent circles of a radii $a=15$, $b=17$, and $c=19$.

I. **Solution by the PROPOSER; J. F. W. SCHEFFER, A. M., Hagerstown, Maryland; A. H. BELL, Hillsboro, Illinois; and F. P. MATZ, M. Sc., Ph. D., Mechanicsburg, Pennsylvania.**

The problem has two cases: first, when the three given circles are tangent internally to the required circle, as in the problem; and, second, when the required circle is tangent to them externally. But *one solution* involving the resolution of a quadratic equation, will give the answers to both cases. We give the figure for the *first* case only.

Join the centers of the three given circles forming the triangle ABC. Put $AF=a=15$ ft., $CH=b=17$ ft., and $BG=c=19$ ft. Draw CE perpendicular to AB. Let O be the center of the required fourth tangent circle. Draw the radii $R=OCH=OAF=OBG$. Drop on CE the perpendicular OO', and on AB the perpendicular OD. We have $AC=a+b$, $AB=a+c$ and $BC=b+c$. It is evident that $AO=R-a$, $BO=R-c$, and $CO=R-b$. We have in the triangle ABC, by geometry: $AB:BC+AC::BC-AC:BE-AE$; that is, $c+a:a+2b+c::c-a:(a+2b+c)(c-a)=BE-AE$.



$\therefore$ $AE=(a^2+ac+ab-bc)\div(c+a)$, and $BE=(c^2+ac+bc-ab)\div(c+a)$. Again in the triangle AOB, we have $AB:AO+BO::AO-BO:AD-BD$; that is, $c+a:2R-(c+a)::c-a:(2R-a-c)(c-a)\div(c+a)=AD-BD$. $\therefore$ $AD=(a^2+ac+Rc-Ra)\div(c+a)$, and $BD=(c^2+ac-Rc+Ra)\div(c+a)$. Now $OO'=DE=BE-BD=(bc-ab+Rc-Ra)\div(c+a)$; also $EO'=DO=\sqrt{(BO^2-BD^2)}=2\sqrt{(R^2ac-Rac^2-Ra^2c)}\div(c+a)$. The perpendicular $CE=2\sqrt{[abc(a+b+c)]}\div(c+a)$. Now $CO'=CE-EO'=\{2\sqrt{[abc(a+b+c)]}-2\sqrt{(R^2ac-Rac^2-Ra^2c)}\}\div(c+a)$. Again $CO=\sqrt{(CO'^2+OO'^2)}=\sqrt{\{4a^2bc+4ab^2c+4abc^2+4R^2ac-4Rac^2-4Ra^2c+(bc-ab+Rc-Ra)^2-8\sqrt{[abc(a+b+c)(R^2ac-Rac^2-Ra^2c)]}\}}\div(c+a)$. Putting this value of $CO=R-b$ and clearing the resulting equation from radicals by two successive involutions, we obtain the quadratic equation, after dividing by $(c+a)^2$; $[2abc(a+b+c)-(a^2b^2+a^2c^2+b^2c^2)]R^2-2abc(ab+ac+bc)R=a^2b^2c^2$.

Whence by resolution,

$$R=\frac{abc(ab+ac+bc)\pm2abc\sqrt{abc(a+b+c)}}{2abc(a+b+c)-(a^2b^2+a^2c^2+b^2c^2)}=\frac{4181235\pm4941901\sqrt{95}}{243611}$$

$=36.93594828+$feet, or $-2.60880378+$ft. The negative value of R, regarded as a positive quantity, is the radius of the circle that is tangent to the three given circles *externally*.

II. Solution by G. B. M. ZERR, A. M., Ph D., Vice President and Professor of Mathematics in the Texarkana College, Texarkana, Arkansas.

Let $BG=a,=19$, $CH=b=17$, $AF=c=15$, $OG=OH=OF=r$.

$$\therefore \cos BCA=\frac{b^2+ab+bc-ac}{b^2+ab+bc+ac}=\frac{97}{192},$$

$$\cos BCO=\frac{b^2+ab+ar-br}{ar+br-b^2-ab}=\frac{306+r}{18r-306},$$

$$\cos ACO=\frac{b^2+bc+cr-br}{cr+br-b^2-bc}=\frac{272-r}{16r-272}.$$

But $\cos BCA=\cos(BCA+BCO)$.

$$\therefore\ r=\frac{abc\{ab+ac+bc\pm2\sqrt{abc(a+b+c)}\}}{4abc(a+b+c)-(ab+ac+bc)^2}=\frac{\{863\pm102\sqrt{95}\}4845}{243611};$$

$\therefore\ r=36.93595$ or -2.608803.

The first value is the radius of circumscribing circle, the second is the radius of the circle inscribed in the space enclosed by the three circles.

PROBLEMS.

50. **Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.**

Draw a line perpendicular to the base of a triangle dividing the triangle in the ratio of m to n.

51. **Proposed by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in the Texarkana College, Texarkana, Arkansas.**

To construct a trapezoid; given the bases, the perpendicular distance between the bases and the angle formed by the diagonals.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him

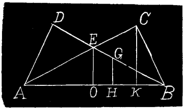
SOLUTIONS OF PROBLEMS.

36. **Proposed by H. C. WHITAKER, B. Sc., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.**

A cube is revolved on its diagonal as an axis. Define the figure described and calculate its volume.

Solution by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in the Texarkana College, Texarkana, Arkansas.

The surface of the solid is that generated by revolving $ADECB$ about AB as an axis.



Let $AD=BC=a$ be the side of the cube. Then $AC=BD=a\sqrt{2}$, $AB=a\sqrt{3}$, $OC=\frac{a\sqrt{3}}{2}$. $AC^2=AB.AK$. $\therefore\ AK=\frac{2a}{\sqrt{3}}$. $CK^2=AK.BK=AK(AB-AK)$.

$$\therefore\ CK=\frac{a\sqrt{2}}{\sqrt{3}}.$$

$$GH=\tfrac{1}{3}CK=\frac{a\sqrt{2}}{3\sqrt{3}}.\qquad EO:AO=CK:AK.\quad \therefore\ EO=\frac{a\sqrt{6}}{4}.$$

Area ACB=area $ADB=\frac{1}{2}AB.CK=\frac{a^2\sqrt{2}}{2}=A$.

Area $AEB=\frac{1}{2}AB.EO=\frac{3a^2\sqrt{2}}{8}=A'$.

V=required volume$=2(2\pi GH.A)-2\pi EO.\frac{A'}{3}$.

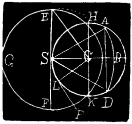
$$\therefore\ V=\frac{4a^3\pi}{3\sqrt{3}}-\frac{a^3\pi\sqrt{3}}{8}=\frac{23a^3\pi}{24\sqrt{3}}=\frac{23a^3\pi\sqrt{3}}{72}.$$

37. **Proposed by J. A. CALDERHEAD, Superintendent of Schools, Limaville, Ohio.**

A man ties two mules—one to the outside of a circular wall, the other to the inside. If the lengths of the ropes of each is one-fourth the circumference of the wall, and both together can graze over one acre of ground: find the circumference of the wall.

I. **Solution by COOPER D. SCHMITT, Professor of Mathematics in the University of Tennessee, Knoxville, Tennessee.**

Let S be the point where the mules are fastened. The mule grazing on the outside, grazes over the semi-circle $EGF+HSK$ and FSP. The mule on the inside grazes over the segment $SHA+$ segment $SKD+$ sector ASD.



We must find the area of these different portions and their sum equals one acre or 160 sq. rods.

1. To find the area of sector $ABDSA$. Since $AS=\frac{\pi r}{2}$, we have $\cos ASB=\frac{1}{4}\pi$. $\therefore\ ASB=38°\ 15'$, and area of sector $=\frac{76\frac{1}{2}}{360}.\pi.\frac{\pi^2r^2}{4}$.

2. To find area of segment SHA. $\angle SCA=103°\ 30'$. Segment SHA=sector $SHAC-\triangle ASC$ $=\frac{103\frac{1}{2}}{360}\pi r^2-\frac{1}{2}r^2\sin 103\frac{1}{2}°$. Segment DSK=segment SAH.

3. Area of $EGF=\frac{1}{2}\pi\left(\frac{\pi^2r^2}{4}\right)$.

4. Area of involute $ESHA$=area CHE+area ECS−area SCH $=$(by the Calculus)$\frac{SC^3}{6r}+\frac{1}{2}SC.r-\frac{1}{4}\pi r^2$. But $SC=\frac{1}{2}\pi r$. $\therefore$ involute

$ESHA=(\frac{1}{2}\pi r)^3\div 6r+\frac{1}{2}.\frac{1}{2}\pi r.r-\frac{1}{4}\pi r^2=\frac{1}{48}\pi^3r^2$.

The area of involute $FSLK$=area of involute $ESHA$.

Combining these different areas, we have

$$r^2\left[\frac{153}{720}\frac{\pi^3}{4}+\frac{207}{360}\pi-\sin 103\tfrac{1}{2}°+\frac{\pi^3}{8}+\frac{n^3}{24}\right]=160.$$

$$r^2\left[\frac{51\pi^3}{960}+\frac{207}{360}\pi-.9724+\frac{\pi^3}{6}\right]=160,$$

$$r^2\left[\frac{211}{960}\pi^3+\frac{207}{360}\pi-.9724\right]=160,$$

$r^2[6.8148+1.8064-.9726]=160$, $r^2(7.6486)=160$, $r^2=20.91$, $r=4.57$. $2\pi r=28.714+$ the circumference of the wall.

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in the Texarkana College, Texarkana, Arkansas.**

Let ASD, in Fig. above, be the given wall. S the point where the mules are fastened, a=radius of wall, $\varphi=\angle KCL$, $PL=\rho$, =radius of curvature of involute KPF, $\theta=\angle ASC$.

We now have the three areas to find:— (1) area $SHABDKS$, (2) the the two equal involute areas SHE and SKF, (3) area semi-circle EGF. $SA=SE=\frac12\pi a$.

$\therefore$ Area semi-circle $EGE=\frac18\pi^3a^2\ldots.(1)$.

Area of an element between two consecutive radii of curvature is $dA=\frac12\rho^2d\varphi=\frac12a^2\varphi^2d\varphi$, since $\rho=a\varphi$.

$$\therefore \text{ Area } (SHE+SKF)=a^2\int_0^{\frac{\pi}{2}}\varphi^2d\varphi=\tfrac{1}{24}\pi^3a^2\ldots.(2).$$

Area common to both circles$=a^2(\pi+2\theta\cos2\theta-\sin2\theta)$, but $2a\cos\theta=\frac12a\pi$, $\therefore$ $\cos\theta=\frac14\pi$.

$\therefore$ Area common to both circles

$$=a^2\left\{\pi+\tfrac14(\pi^2-8)\cos^{-1}\frac{\pi}{4}-\frac{\pi}{8}\sqrt{16-\pi^2}\right\}\ldots.(3).$$

$$\therefore a^2\left\{\tfrac16\pi^3+\pi+\tfrac14(\pi^2-8)\cos^{-1}\frac{\pi}{4}-\frac{\pi}{8}\sqrt{15-\pi^2}\right\}=160 \text{ sq. rods.}$$

$$\therefore a^2\left(7.337+.4674\cos^{-1}\frac{\pi}{4}\right)=160 \text{ sq. rods.}$$

$\therefore$ $7.64896a^2=160$, $a=4.5736$ rds.

$2\pi a=28.7368$ rods, =circumference required.

Also solved by *A. H. BELL* and *F. P. MATZ*.

PROBLEMS.

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45. Proposed by Dr. GEORGE LILLEY, Portland, Oregon.

A fly starts from a point in the circumference of a table, 3 feet in diameter, and travels uniformily along the diameter to a point in the circumference of the table directly opposite the starting point. The table moves uniformily to the right about a center axis in such manner that it makes one complete revolution while the fly passes over its diameter. Find the absolute path described by the fly and the ratio of rates of movement of the table and the fly.

46. Proposed by H. C. WHITAKER, M. E., Sc. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

There are four points A, B, C, and D in space. Point D remains fixed with its co-ordinates (1, 2, 2) feet. At a given time A is at (2, 3, 4) feet, is moving in a straight line at the rate of 3 feet per minute, and has passed through (5, 9, 10) feet: B is at (1, 4, 2) feet, moves in a straight line at the rate of 7 feet per minute, and will pass through (−2, 2, 8) feet; C is at the origin and moves along the axis of X in the direction of x positive at the rate of 6 feet per minute.

The motion of the points being continuous before and after the given time, required the times when the volume of the tetrahedron whose edges are the lines joining these points will be 108 cubic inches.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

26. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

If an elastic sphere be electrified in such a manner that the initial internal pressure remains constant, determine an expression for the *ratio of the electrical densities* when the volume of the sphere has been increased to $(m+1)$ times its initial volume.

I. Solution by the PROPOSER.

Assuming r as the *initial* radius, and R as any *greater* radius, we have $x^2+y^2+z^2=r^2$ ....(1) and $x^2+y^2+z^2=R^2$ ....(2), as the result of the *volume-increase*. These Cartesian equations are transformable by means of the well-known equations, $x=\rho\cos\theta$ and $y=\rho\sin\theta$. Choosing the integral limits so as to give the total internal pressure, we have for the work done in increasing the initial sphere, radius r, to a sphere of radius R, when $z_1=\surd(r^2-\rho^2)$ and $z_2=\surd(R^2-\rho^2)$,

$$W=8\left(\frac{2T}{r}\right)\int_0^{\frac{1}{2}\pi}\int_0^r 2\rho d\theta d\rho\times 8\int_0^{\frac{1}{2}\pi}\left[\int_0^R\int_0^{z_2}\rho d\rho dz-\int_0^r\int_0^{z_1}\rho d\rho dz\right]d\phi$$

$$=\tfrac{4}{3}\pi(8\pi Tr)[R^3-r^3]\ldots.(3).$$

Specializing in (3), for $R^3=mr^3$ and for $R^3=(m+1)r^3$; representing the electrical densities, the ratio of which is to be determined, by $\triangle_m$ and $\triangle_{m+1}$; then equating the specialized results obtained from (3), to the potential energies of the electrifications similarly specialized,—that is, according to Helmholtz's formula, to

$$\frac{Q}{2r}=\left(\frac{\triangle^2_R}{2r}\right)\left[8\int_0^{\frac{1}{2}\pi}\int_0^r 2\rho d\theta d\rho\right]^2,$$

we have the required ratio,

$$\frac{\triangle_m}{\triangle_{m+1}}=\sqrt{\left(1-\frac{1}{m^2}\right)}\ldots.(4).$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in the Texarkana College, Texarkana, Arkansas.

Let p be the excess of pressure, on the assumption that the external pressure is costant.

Let u=original volume, then when the sphere is n times its original volume, the work done in this electrification must be

$p(n-1)u$, (see *Minchin's Statics*).

Let v be the potential, Q the charge of this electrification, r=radius of sphere, σ=electrical density.

$\therefore$ energy of electrification$=\frac{1}{2}vQ$. But $v=\dfrac{Q}{r}$ and $Q=4\pi r^2\sigma$.

$\therefore p(n-1)u=\frac{1}{2}\dfrac{Q^2}{r}=8\pi^2r^3\sigma^2$. But $\frac{4}{3}\pi r^3=nu$.

$\therefore p(n-1)u=6n\pi\sigma^2u$.

$\therefore p(n-1)=6n\pi\sigma^2$.

Similarly, when the sphere becomes $(m+1)$ times its original size we get, if σ_1 is the density, $pm=6(m+1)\pi\sigma_1^2$.

$$\therefore \frac{\sigma}{\sigma_1}=\sqrt{\frac{(m+1)(n-1)}{mn}}=\sqrt{\frac{m^2-1}{m^2}}\text{ if } n=m.$$

27. Proposed by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in the Texarkana College, Texarkana, Arkansas.

One thousand balls, each having a mass of 10 grams, and each moving with a velocity of 10 kilometers per second, are confined in a certain space with elastic walls. Into the same space are now introduced one thousand balls each of 1000 grams mass, and moving with a velocity each of 10 kilometers per second: collisions take place, and finally, after a number of encounters, the average kinetic energy of each of the two thousand balls is the same. Show that this is 5.75(10)11 in the centimeter-gram-system.

Solution by O. W. ANTHONY, Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland and the PROPOSER.

By Avogadro's hypothesis and kinetic theory, if M, M_1 are the masses and V, V_1 the velocities of two balls, then $\frac{1}{2}MV^2$ and $\frac{1}{2}M_1V_1^2$ are their respective kinetic energies. As the average kinetic energy of each ball is equal, we get $\frac{1}{2}MV^2=\frac{1}{2}M_1V_1^2$.

$\therefore$ The average kinetic energy$=\frac{1}{2}(\frac{1}{2}MV^2+\frac{1}{2}M_1V_1^2)=E$.

$\therefore E=\frac{1}{4}(MV^2+M_1V_1^2)$.

Let $V=V_1$, then $E=\frac{1}{4}(M+M_1)V^2$.

V=10 kilometers$=10^6$ centimeters.

M=10 grams, M_1=100 grams.

$\therefore E=\frac{1}{4}(10+100)10^{12}=\frac{11}{4}\times10^{13}=2.75\times10^{13}=2.75(10)^{13}$.

Also solved by *F. P. MATZ*.

PROBLEMS.

33. Proposed by F P. MATZ, M. Sc , Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania

At what angle with the axis of a stalk must a *sharp* wedge-shaped blade be struck, in order to sever the stalk with the *least* force?

34. Proposed by O. W. ANTHONY, Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland

A particle is placed within a thin cylindrical shell without ends. Find the resultant attraction, the cylinder being composed of matter attracting according to the laws of nature.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

25. Proposed by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in Texarkana College, Texarkana, Arkansas.

The probability that the distance of two points taken at random in a given convex area, A, shall excel a given limit (a) is

$$\triangle=\frac{1}{3A^2}\int\int(C^3-3aC+2a^3)dpd\theta,$$

where C is a chord of the area, whose co-ordinates are p θ; the integration extending to all values of p, θ, which give a chord $C>a$. What is $\triangle$ when the area is a circle? If in the circle $a=r=$radius, $\triangle=\frac{3\sqrt{3}}{4\pi}$.

Solution by the PROPOSER.

Let ST be the chord; P, Q the random points, $OR=p$, the perpendicular, $\angle ROA=\theta$, $SQ=x$, $PQ=y$, $ST=C$.

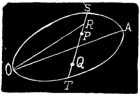
An element of area at Q is $dpdx$; at P, $yd\theta dy$. The limits of x are o and $C-a$; of y, $C-x$ and a, and doubled.

Since two points can be taken in A^2 ways in the area A, we get for the required chance

$$\triangle=\frac{2}{A^2}\int\int d\theta dp\int_o^{C-a}\int_a^{C-x}ydxdy,$$

$$=\frac{1}{A^2}\int\int d\theta dp\int_o^{C-a}(C^2-2Cx+x^2-a^2)dx,$$

$$=\frac{1}{3A^2}\int\int(C^3-3a^2C+2a^3)d\theta dp.$$



Now let the area be a circle with the origin at centre. Then $C=2\sqrt{r^2-p^2}$, when r=radius. The limits of θ are 0 and $\frac{1}{2}\pi$, doubled, of p, 0 and $\frac{1}{2}\sqrt{4r^2-a^2}$, and doubled.

$$\therefore\ \triangle=\frac{4}{3\pi^2r^4}\int_o^{\frac{1}{2}\pi}\int_o^{\frac{1}{2}\sqrt{4r^2-a^2}}\{8(r^2-p^2)^{\frac{3}{2}}-6a^2(r^2-p^2)^{\frac{1}{2}}+2a^3\}d\theta dp.$$

$$=\frac{1}{2\pi^2r^4}\left\{(a^3+2ar^2)(4r^2-a^2)^{\frac{1}{2}}+8r^2(r^2-a^2)\sin^{-1}\left(1-\frac{a^2}{4r^2}\right)^{\frac{1}{2}}\right\}\int_o^{\frac{1}{2}\pi}d\theta$$

$$=\frac{1}{4\pi}\cdot\frac{a}{r}\left(2+\frac{a^2}{r^2}\right)\left(4-\frac{a^2}{r^2}\right)^{\frac{1}{2}}+\frac{2}{\pi}\left(1-\frac{a^2}{r^2}\right)\sin^{-1}\left(1-\frac{a^2}{4r^2}\right)^{\frac{1}{2}}.$$

If $a=r$, $\triangle=\dfrac{3\sqrt{3}}{4\pi}$.

26. Proposed by J. WATSON, Middlecreek, Ohio.

Find the average area of all right-angled triangles having a given hypotenuse.

I. Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Let h=the given hypotenuse, and x=the base; then will $\sqrt{(h^2-x^2)}$= the perpendicular, and the area of the triangle is $A=\frac{1}{2}x\sqrt{(h^2-x^2)}$. Hence the required average area becomes, if $\frac{1}{2}h\sqrt{2}=a$,

$$A=\int_o^a Adx\div\int_o^a dx,=\tfrac{1}{12}h^2(2\sqrt{2}-1).$$

Second Solution.

Represent the base by $h\cos\theta$, and the perpendicular by $h\sin\theta$; then we have $A=\frac{1}{2}h^2\int_o^{\frac{1}{4}\pi}\sin\theta\cos^2\theta d\theta\div h\int_o^{\frac{1}{4}\pi}\cos\theta d\theta,=\frac{1}{12}h^2(2\sqrt{2}-1)$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Vice President and Professor of Mathematics in Texarkana College, Texarkana, Arkansas; O. W. ANTHONY, Professor of Mathematics, New Windsor College, New Windsor, Maryland; J. F. W. SCHEFFER, A. M., Hagerstown, Maryland; and H. W. DRAUGHON, Olio, Mississippi.

Let $AC=2a$=hypotenuse of triangle, $AD=DC=DB=a$, and $\angle CDB=\theta$. $\therefore\ BE=a\sin\theta$.

$\therefore$ Area $=a^2 \sin\theta$. Perimeter$=2a(\cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta+1$.
Let A=average area, P=average perimeter.

$$\therefore A=\frac{a^2\int_0^\pi \sin\theta d\theta}{\int_0^\pi d\theta}=\frac{2a^2}{\pi}.$$

$$P=\frac{2a\int_0^\pi(\cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta+1)d\theta}{\int_0^\pi d\theta}=\frac{2a(4+\pi)}{\pi}.$$

III. **Solution by O. W. ANTHONY, Professor of Mathematics, New Windsor College, New Windsor, Maryland; P. S. BERG, Larimore, North Dakota; and H. W. DRAUGHON, Olio, Mississippi.**

Let a=the hypotenuse, and x one of the sides.

Then the area of the triangle $=\frac{1}{2}\times\sqrt{(a^2-x^2)}$ and the required average area

$$=\frac{\int_0^a \frac{1}{2}x\sqrt{(a^2-x^2)}}{\int_0^a dx}=\frac{a^2}{6}.$$

NOTE.—We have published these various solutions in order that the authors may compare their results and decide upon some definite method of solving this problem. It is our opinion that the result, $\frac{a^2}{2\pi}$, is correct; for the number of triangles is equal to the semi-circumference whose diameter is the given hypotenuse a, that is to say, the number of triangles is proportional to the locus of the vertex of the right angle and not proportional to the variable sides. But if this method of solution is adopted for this problem, it will vitiate the solutions of a great many problems in *Average* and *Probability*,—solutions that have gone in print in numerous Journals and text books.

Dr. Artemas Martin proposed this problem in the *Educational Times*, London, England, for October, 1869. The published solutions both in the *Times* and the *Reprint* give the answer $\frac{a^2}{2\pi}$. Dr. Martin says, *Mathematical Magazine*, Vol I., p. 216, "I do not regard that method [the method assuming that the vertices of the right angle are uniformly distributed on the semi-circumference of a circle whose diameter is a] as correct. The vertices of the right angle will all be situated on a semi-circumference whose diameter is a, but they will *not* be uniformly distributed on it. In order to obtain *all* the triangles, one of the legs should be made to vary uniformly from 0 to a."

He then produces a very beautiful solution without the aid of the calculus and gets as a result, $\frac{1}{6}a^2$. Then he gives another solution which is the same as III. above.

Now it seems to us that whether the triangles are uniformly distributed on the semi-circumference or not is of no concern in the solution of the problem. The question is (1), how many right triangles are there whose hypotenuses are a; and (2), what is the area of each one of these triangles? Having found the numbers answering to these questions, we divide the sum of the areas of the triangles by the number of triangles, according to the principle of *Mean Value*, and get the required result. The *sum* of the areas of the triangles is easily found by the aid of the Calculus and the number of triangles is equal to the semi-circumference of a circle whose diameter is a. This is, in our opinion, the correct solution and agrees with II. above. All of the above solutions are, doubtless, correct from the stand-points of the authors, but the stand-points of some must be wrong. As it is the object of the MONTHLY to aid in the establishment of sound principles in all departments of Mathematics, we shall be pleased to publish, in the next issue, brief notes on these solutions from various contributors. [EDITOR.]

PROBLEMS.

33. Proposed by F. P. MATZ, M. Sc, Ph. D, Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the average area of all regular polygons having a *constant* apothem.

34. Proposed by B. F. FINKEL, A. M., Professor of Mathematics, Drury College, Springfield, Missouri.

Two points are taken at random on the circumference of a semi-circle. Find the chance that their ordinates fall on either side of a point taken at random on the diameter.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

DIOPHANTUS' EPITAPH.

Hic Diophantus habet tumulum, qui tempora vitae
Illius mira denotat arte tibi,
Egit sextantem juvenis; languine malas
Vestire hinc coepit parte duodecima.

Septante uxori post haec sociatur, et anno
Formosus quinto nascitur inde puer.

Semissem aetatis postquam attigit ille paternae
Infelix subita morte peremptus obit.
Quatuor aestates genitor lugere superstes
Cogitur: hinc annos illius assequere.

An Equation for the "Sum of Squares equal a Square" by R. J. ADCOCK, Larchland, Illinois.

The following identical equation for the sum of squares=a square, I have not seen published. If $u=x+y+z+v+w$, $u^2=x^2+y^2+z^2+v^2+w^2+2xy+2xz+2xv+2xw+2yz+2yv+2yw+2zv+2zw+2vw$; and if the sum of products two in a set=0, $u^2=x^2+y^2+z^2+v^2+w^2$, $w=-\dfrac{xy+xz+xv+yz+yv+zv}{x+y+z+v}$,

$$u^2=x^2+y^2+z^2+v^2+\left(\frac{xy+xz+xv+yz+yv+zv}{x+y+z+v}\right)^2=$$

$$\left[x+y+z+v-\left(\frac{xy+xz+xv+yz+yv+zv}{x+y+z+v}\right)\right]^2.$$

Clearing of fractions and reducing, $[x(x+y+z+v)]^2+y^2(x+y+z+v)^2+z^2(x+y+z+v)^2+v^2(x+y+z+v)^2+(xy+xz+xv+yz+yv+zv)^2=(x^2+y^2+z^2+v^2+xy+xz+xv+yz+yv+zv)^2$. True for three or any greater number of letters.

COMMENT.—In the solution of problem 21, page 163, Vol. II, May No., Dr. Martin uses an ingenuous method for finding a general formula "to find nine integral square numbers whose sum is a square number."

The same formula, expressed for finding n integral square numbers whose sum is a square number, may be produced, more directly, from $(2pq)^2+(p^2-q^2)^2=(p^2+q^2)^2$. Put $p^2=m_1^2+m_2^2+m_3^2+\dots+m^2_{n-1}$ and $q^2=m^2_n$, in which $m_1, m_2, m_3, \dots m_n$ represent any n integers.

We readily obtain $(2m_1m_n)^2+(2m_2m_n)^2+(2m_3m_n)^2+\dots+(2m_{n-1}m_n)^2+(m_1^2+m_2^2+m_3^2+\dots+m^2_{n-1}-m^2_n)^2=(m_1^2+m_2^2+m_3^2+\dots+m^2_n)^2$.

Illustration. Let $n=9$, and put $m_1=1$, $m_2=2$, $m_3=3$, $m_4=4$, $m_5=5$, $m_6=6$, $m_7=7$, $m_8=8$, and $m_9=9$. Substituting these values in the formula and dividing by 12, we obtain $1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+14^2=20^2$.

PROBMEMS.

37. Proposed by A. H. BELL, Hillsboro, Illinois.

Find the first four, integral values of n in $\dfrac{n(5n-3)}{2}=\square$.

This is the general form of septagonal numbers, 1, 7, 18, 34, 55, etc.

38. Proposed by H. C. WILKES, Skull Run West Virginia.

Let n be any number and let $n^3+1=x$. Then $x^3+(2x-3)^3+(nx-3n)^3=n^3x^3$. How can this be demonstrated; it will always be found true on trial.

39. Proposed by F. P. MATZ, M. Sc., Ph. D, Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The mth root of the nth power of an *integral* number is a perfect pth power. What is the number?

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

16 Yale Senior Prize Problem.---Contributed by H. A. NEWTON, LL. D., Professor of Mathematics, Yale University, New Haven Connecticut.

The axes of two right cylinders whose bases are circles of 4 and 6 inches radius respectively, intersect at right angles. Compute to four decimal places the lengths of the *curves* of intersection of the two surfaces.

Solution by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Make $m=6$ inches, and $n=4$ inches; then the Cartesian equations of the cylinders become $z^2+x^2=m^2 \ldots (1)$ and $y^2+x^2=n^2 \ldots (2)$.

$$\therefore \frac{dz}{dx}=-\frac{x}{z}=-\frac{x}{\sqrt{(m^2-x^2)}} \ldots (3),$$

$$\text{and } \frac{dy}{dx}=-\frac{x}{y}=-\frac{x}{\sqrt{(n^2-x^2)}} \ldots (4).$$

Hence the expression for the *lengths* of the *curves* of intersection of the two surfaces becomes, *Todhunter's Integral Calculus*, p. 116,

$$\mathsf{L}=8\int_0^n\sqrt{\left(1+\frac{x^2}{m^2-x^2}+\frac{x^2}{n^2-x^2}\right)}dx \ldots (5).$$

Make $c^2=n^2/m^2$, and $x=n\sin\phi$; then $dx=n\cos\phi d\phi$. Transforming (5), etc.,

$$\mathsf{L}=8\int_0^n\sqrt{\left(\frac{m^2n^2-x^4}{(m^2-x^2)(n^2-x^2)}\right)}dx=8n\int_0^{\frac{1}{2}\pi}\sqrt{\left(\frac{1-c^2\sin^4\phi}{1-c^2\sin^2\phi}\right)}d\phi$$

$$=8n\int_0^{\frac{1}{2}\pi}[1+\tfrac{1}{2}c^2\sin^2\phi+(\tfrac{3}{8}c^4-\tfrac{1}{2}c^2)\sin^4\phi+(\tfrac{5}{16}c^6-\tfrac{1}{4}c^4)\sin^6\phi+\text{etc.}]d\phi$$

$$=4\pi n\left[1+\frac{c^2}{16}+\frac{111c^4}{4016}+\text{etc.}\right]=51.9363+\text{inches}.$$

20. Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D., Penn Yan, Yates County, New York

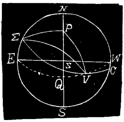
When will the Dog-Star and the Sun *rise together* in lanitude $\lambda=+42°30'$, if the right Ascension of the said star be $\alpha=6h.\ 40m.\ 30s.$ and the Declination $\delta=-16°\ 33'\ 56''$?

Solution by F. P. MATZ, Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

According to *Chauvenet's Spherical and Practical Astronomy*, Vol. I.,

218; Art. 153, we have for the *hour-angle* of the Dog-Star, $t=\pm\cos^{-1}[-\tan\lambda\tan\delta], =74°\ 10'\ 57.''77, =4h.\ 56m.\ 43.851s.$ The *sidereal* time of the rising of the Dog-star is, therefore, $\Theta=\alpha-t, =1h.\ 43m.\ 46.149s., =25°\ 56'\ 32''.325.$

The problem now is to find on what day of the year the *Sun* will rise at the sidereal time of the rising of the Dog-Star.



In the accompanying diagram, let Σ represent the Sun just rising; $\angle VPQ=\Theta$; $QZ=\lambda$; $ZP=90-\lambda$; $VP=Z\Sigma=90°$; $\angle PVE=90°$; and $\angle\Sigma VE=23°\ 27'\ 18''.82$. Now. $\angle PV\Sigma=66°\ 32'\ 41''.18$; and from the spherical triangle PVZ, we deduce $\angle PVZ=25°\ 32'\ 24''$, and $VZ=48°\ 30'\ 8''$. Obviously $\angle ZV\Sigma=90°-(\angle PVZ+\angle\Sigma VE)=41°\ 0'\ 17''.18$. From the spherical triangle $VZ\Sigma$, we deduce $\angle V\Sigma Z=29°\ 26'\ 18''$, and $V\Sigma=131°\ 25'\ 28''.4$. Now, $V\Sigma$ is the *longitude* of the Sun at the time the Dog-Star and the Sun rise together. With a mean daily motion of $59'\ 8''.3302$, the Sun reaches this longitude $133\frac{1}{3}$ days immediately succeeding the *twenty-first* of March, or on the *first* day of August.

This problem can be solved in *four* different ways; and some of the results of these solutions indicate the *second* day of August as the required date.

22. Proposed by F. P. MATZ, D. Sc, Ph. D, Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

From what kind of dry wood must a ship's log be cut, in order that the log may float with its center of gravity at the water's surface?

I Solution by B. F. BURLESON, Oneida Castle, New York, and the PROPOSER.

The ship's log, a *circular sextant* in form, floats with the *convex* part immersed. If the central semi-angle be α, the portion of the log *not* immersed is αr^2. Consequently, the portion of the log immersed, becomes,

$$A=(\alpha-4\sin^2\alpha\tan\alpha/9\alpha^2)r^2\ \ldots\ (1).$$

If $\dfrac{1}{G}$=the specific gravity of the log,

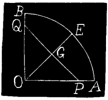
$$\frac{1}{G}=\frac{9\alpha^3-4\sin^2\alpha\tan\alpha}{9\alpha^3}=1-\frac{8\sqrt{3}}{\pi^3}=.5531096,$$

which, according to *Hutton's* Table of Specific Gravity of Bodies, is the specific gravity of Dry *Fir*; and, according to *Scribner's* Table of the Specific Gravity of Bodies, is the specific gravity of Dry *White Pine*.

II Solution by G. B. M. ZERR, A. M., Ph. D, Vice President and Professor of Mathematics and Sciences in the Texarkana College, and Conservatory of Music, Texarkana, Arkansas.

Taking the "log" in the shape of a quadrant of a circle and supposing it to float with the arc down, let G be the center of gravity ρ=density of the

wood, 1.026 the density of the sea-water. Then $OG=\frac{4r\sqrt{2}}{3\pi}$, $OP=OQ=\frac{8r}{3\pi}$.



$\therefore$ area $POQ=\frac{32r^2}{9\pi^2}$, area quadrant$=\frac{1}{4}\pi r^2$, area

$$QBEAPQ=\tfrac{1}{4}\pi r^2-\frac{32r^2}{9\pi^2},\quad \therefore\ 1.026\left\{\tfrac{1}{4}\pi r^2-\frac{32r^2}{9\pi^2}\right\}=\tfrac{1}{4}\pi r^2\rho.$$

$$\therefore\ \rho=1.026\left(1-\frac{128}{9\pi^3}\right)=.5554.$$

This is the density of Juniper tree (dry) and very nearly the density of white pine (.554).

23. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics in Texarkana College, Texarkana, Arkansas.

Pliny says, "Thales determined the cosmical setting of the Pleiades to have happened in his time 25 days after the vernal equinox". Determine the time when Thales lived from the following data:—Latitude of Miletus 37° 30′, the precession of the equinox 50″.34 anually, the R. A. of Alcyon (η Tauris) Jan. 1, 1895, 3h. 41m. 15 sec. declination 23° 46′ 49″ N.

Solution by the PROPOSER.

Let λ=latitude of Miletus, a, δ, t, a_1, δ_1, t_1, the R. A. declination and hour-angle of Alcyon and the Sun respectively; ε=the obliquity of the ecliptic, ω=the distance the Sun has traveled on the ecliptic after the vernal equinox.

Then $\cos t=-\tan\lambda\tan\delta\ldots.(1)$. $\cos t_1=-\tan\lambda\tan\delta_1\ldots.(2)$. $\sin a_1=\tan\delta_1\cot\varepsilon\ldots.(3)$. $a_1+t_1=a+t=\theta$, or $a_1=\theta-t_1\ldots.(4)$. $\sin a_1=\sin(\theta-t_1)\ldots.(5)$. From (3) and (5), $\sin(\theta-t_1)=\tan\delta\cot\varepsilon\ldots.(6)$. From (2) and (6), $\tan\delta_1=\frac{\sin(\theta-t_1)}{\cot\varepsilon}=-\frac{\cos t_1}{\tan\lambda}\ldots.(7)$. From (7)

$$\tan t_1=\frac{\sin\theta\tan\lambda+\cot\varepsilon}{\cos\theta\tan\lambda}\ldots.(8).$$

Also $\cot\omega=\cos\varepsilon\cot a_1\ldots.(9)$. Now $\lambda=37°\ 30'$, $\delta=23°\ 46'\ 49''$, a=3h. 41m. 15 sec., $\varepsilon=23°\ 27'\ 13''$. From (1), $t=109°\ 45'\ 43''.57$=7h. 19m. 2.9 sec. $a+t=\theta$=11h. 0m. 17.91 sec.$=165°\ 4'\ 28''.57$. From (8) $t_1=106°\ 30'\ 10''.94$ =7h. 6m. 0.73 sec. From (4), a_1=3h. 54m. 17.18 sec.$=58°\ 34'\ 17''.7$. From (9), $\omega=60°\ 43'\ 28''.47$.

In one day the Sun moves $59'\ 8''.35$. $(59'\ 8''.35)\times25=24°\ 38'\ 28''.75$. $60°\ 43'\ 28''.47-24°\ 38'\ 28''.75=36°\ 4'\ 59''.72=129899''.72$. $129899''.72\div50''.34=2580.44+$years. $2580.44-1894=686.44$ B. C., when Thales determined the cosmical setting of the Pleiades. $60°\ 43'\ 28''.47\div59'\ 8''.35=61.6085$ days after vernal equinox. $61.6085-25=36.6085$. $59'\ 8''.35\div50''.34=70.48768$ years. $79.48768\times36.6085=2580.448+$. $2580.438-1894=686.448$ B. C.

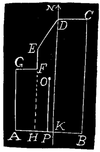
Also solved by *F. P. MATZ.*

24. **Proposed by D. H. DAVISON, C. E., Minonk, Illinois.**

For the purpose of locating the most eligible point for a county-seat, it is required to determine the centre of a county whose dimensions are as follows: Beginning at the S. W. corner, thence E. 15 miles, thence N. $33\frac{2}{3}$ miles, thence W. 6 miles to the north end of the meridian running south through the county, thence southwesterly to a point 6 miles W. from the meridian and $9\frac{2}{3}$ miles S. of the north end of said meridian, thence S 3 miles, thence W. 3 miles,and thence S. 21 miles to the place of beginning.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science in Texarkana College, Texarkana, Arkansas; and F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Let $\bar{x}$, $\bar{y}$, be the co-ordinates of the centroid, and divide the county into three parts as in the figure, then we eaily get with A as origin



$$\bar{x}=\frac{\int_0^3\int_0^{21}xdxdy+\int_3^9\int_0^{19\frac{1}{6}+\frac{22}{18}x}xdxdy+\int_9^{15}\int_0^{33\frac{2}{3}}xdxdy}{\int_0^3\int_0^{21}dxdy+\int_3^9\int_0^{19\frac{1}{6}+\frac{22}{18}x}dxdy+\int_9^{15}\int_0^{33\frac{2}{3}}dxdy}$$

$$=8\tfrac{163}{876}\text{ miles.}$$

$$\bar{y}=\frac{\int_0^3\int_0^{21}ydxdy+\int_3^9\int_0^{19\frac{1}{6}-\frac{22}{18}x}ydxdy+\int_9^{15}\int_0^{33\frac{2}{3}}ydxdy}{\int_0^3\int_0^{21}dxdy+\int_3^9\int_0^{19\frac{1}{6}+\frac{22}{18}x}dxdy+\int_9^{15}\int_0^{33\frac{2}{3}}dxdy}$$

$$=15\tfrac{167}{784}\text{ miles.}$$

∴ Measure east from beginning $8\frac{163}{876}$ miles, then north $15\frac{167}{784}$ miles.

[The proposition that, "The point of the area of a triangle, which has the sum of its distances to all other points of the area a *minimum*, is the centre of gravity of the area," which I think holds for other figures, practically solves problem 24, No. 2. I have made out the proof for the triangle but it occupies two pages. R. J. ADCOCK, Larchland, Illinois.]

Also solved by *P. S. BERG*.

A CORRECTION.—On page 246 of the MONTHLY, my remarks in the lower four lines *above* the *Note*, are not true and should be expunged. They were hastily made upon insufficient investigation. In prob. 20, those remarks hold almost true, but in the general problem they can not ever be true. My solution is not at all affected by those misstatements. The solution may be more easily understood by adding, that, "when Sirius rises, some point of the ecliptic is then rising, and as the Sun is always on the ecliptic the Sun must be at that point, in order to rise synchronously with Sirius.

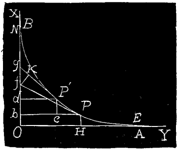
S. H. WRIGHT.

34. **Proposed by GEORGE LILLEY, Ph. D., LL. D., Park School, Portland, Oregon.**

A hare is at O, and a hound at E, 40 rods east of O. They start at the same instant each running with uniform velocity. The hare runs north, The hound runs directly towards the hare and overtakes it at N, 320 rods from O. How far did the hound run?

III. Solution by J. M. BANDY, A. M., Professor of Mathematics in Trinity High School Trinity, North Carolina.

Let $OE=a$, $ON=b$, r=rate of hound, r'=rate of hare, $\frac{r}{r'}=n$, ratio of their speed; co-ordinates of $P=(x, y)$, and $z=EP$. By the theory of curves, $dz=\sqrt{1+\frac{dx^2}{dy^2}}\,dy\ldots(1)$. Now $bf=\frac{ydx}{-dy}$, (y being a decreasing function of x is negative).



$\therefore Of=x+\frac{ydx}{-dy}$. By the problem, $\frac{EP}{x+\frac{ydx}{-dy}}$

$=\frac{r}{r'}=n$, or $EP=n\left(x+\frac{ydx}{-dy}\right)\ldots\ldots(2)$.

$\therefore x+\frac{ydx}{-dy}=\frac{1}{n}\int\sqrt{1+\frac{dx^2}{dy^2}}\,dy\ldots\ldots(3)$. Put $\frac{dx}{dy}=p$, then $dx=pdy$, and we have, $x-py=\frac{1}{n}\int\sqrt{1+p^2}\;dy\ldots(4)$. Differentiating (4) and observing that $dx=pdy$, we have, $-ydp=\frac{1}{n}\sqrt{1+p^2}\,dy\ldots(5)$.

From (5), $\frac{dp}{\sqrt{1+p^2}}=-\frac{dy}{ny}\ldots(6)$. Integrating (6), we have, $\log[p+\sqrt{1+p^2}$ $=-\frac{1}{n}\log y+C=\frac{1}{n}\log\frac{1}{y}+C\ldots(7)$.

When $x=0$, $p=0$, and $y=a$, and $C=\frac{1}{n}\log a$.

$\therefore \log[p+\sqrt{1+p^2}=\frac{1}{n}\log\frac{a}{y}\ldots(8)$. Passing from log. $p+\sqrt{1+p^2}$ $=\left(\frac{a}{y}\right)^{\frac{1}{n}}\ldots(9)$. Solving for p, we have, $2p=\left(\frac{a}{y}\right)^{\frac{1}{n}}-\left(\frac{y}{a}\right)^{\frac{1}{n}}\ldots(10)$. But $p=\frac{dx}{dy}$.

$\therefore 2dx=\left(\frac{a}{y}\right)^{\frac{1}{n}}dy-\left(\frac{y}{a}\right)^{\frac{1}{n}}dy\ldots(11)$. Integrating (11), $2x=\frac{a^{\frac{1}{n}}y^{1-\frac{1}{n}}}{1-\frac{1}{n}}$ $-\frac{y^{1+\frac{1}{n}}}{a^{\frac{1}{n}}\left(1+\frac{1}{n}\right)}+C\ldots(12)$. When $x=0$, $y=a$, and (12) gives val. of $C=\frac{2na}{n^2-1}$.

$$\therefore 2x=\frac{a^{\frac{1}{n}}y^{1-\frac{1}{n}}}{1-\frac{1}{n}}-\frac{y^{1+\frac{1}{n}}}{a^{\frac{1}{n}}\left(1+\frac{1}{n}\right)}+\frac{2na}{n^2-1}\ldots(13.$$

This is the equation of the curve described by the dog, and it is called the "Curve of pursuit." When the dog overtakes the hare, $y=0$, $x=ON=b$.

$\therefore$ (12) becomes, $\dfrac{na}{n^2-1}=b\ldots.(13)$. Solving (13) as a quadratic in n, we

have, $n=\dfrac{1}{2b}(a\pm\sqrt{4b^2+a^2})$, or $n=1.0644+$. Substituting this value of n in (2), remembering that when the dog overtakes the hare, $EP=s$, $x=320$, $y=0$, we have, $s=1.0644\times320=340.624+$ rods.

PROBLEMS.

33. Proposed by Professor ALEXANDER ROSS, C. E., Sebastopol, California.

From a point P without a rectangular field ABC, the distances PA, PB, and PC measured to the corners are, respectively, 70, 40, and 60 chains. What is the area of the field?

34. Proposed by THOS. U. TAYLOR, C. E., M. C. E., Department of Engineering, University of Texas, Austin, Texas.

Given a variable parrallelogram $ABCP$, where P remains fixed. A moves on an irregular plane curve (closed) and C moves on another plane curve (closed) whose plane is parallel to the plane of (A) curve. The generator PC moves completely around and returns to its initial position, AB always moving parallel to PC, and, of course, returns to its initial position. If distances between planes (A) and ($C=h$), show by elementary mathematics and without using theorem of Koppe that volume of solid generated by variable parallelogram $ABCP=\frac{1}{2}h$ (area generated by $AP+$ area generated by BC).

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SPACE.

Space is an entity, outside of the human mind, extended in three directions at right angles to each other, continues, immaterial, immmovable, inflexible and illimitable. It is an entity, sui generis, neither psychical nor physical.

It is cognized but not created by the mind of man and is, doubtless, what it is cognized to be.

A fourth dimension has never been discovered.

Arthur Willink in "The World of the Unseen," pages 90 and 91, locates his hypothetical "Higher Space" in an unknowu direction from our space.

A straight line drawn from us can never reach it and a straight line drawn from the "Higher Space" "towards our space will only pass through one point in our space." For, says he, "If it could pass through more than one point we should know its direction, since two points in a straight line are sufficient to determine the direction of that line."

Principal Campbell in his Philosophy of Rhetoric distinguishes between the "*unintelligible*" and the "*absurd*". Arthur Willink's speculations respecting "Higher Space" are magnificent specimens of the "absurd."

John N. Lyle.

THOSE ASTRONOMICAL CRITICISMS.

To Dr. S. H. Wright's criticisms on page 246 of the July-August Monthly, I beg to make a few remarks.

I. Without wasting any time and energy in the *discussion* of supplementary hour-angles, *my* method of calculation by running the solar system on Dog-Star time, "gets there" far more expeditiously than does Dr. Wright's *admirable* though elaborate and rather unintelligible method.

II. Since I am sixty miles from my copy of *Bartlett's Spherical Astronomy* from which I substantially *borrowed* the sentence relative to the "reversed crescent" illuminated on the "Waning Moon," I am unable to determine whether I *correctly transcribed this sentence* which Dr. Wright so vigorously attacks; but from what I glean from the *Encyclopedia Britannica* and from the astronomical works of Professors Newcomb, Young, etc., I am led to claim: "*Our flag is still there.*"

F. P. Matz.

LAMBERT'S REASON FOR HOLDING THAT THE PARALLEL-AXIOM NEEDS PROOF.

Lambert called by Kant "der unvergleickliche man" is reported as saying that—"the parallel axiom needs a proof, since it does not hold for the geometry on a sphere."

He here assumes without proof that the parallel-axiom does not hold for the geometry on a sphere. That is, he regards this assumption as axiomatic.

Grant that the parallel-axiom does not hold for *spherical* geometry, what has that to do with its holding or not holding for *plane* geometry?

A *plane* surface and a *spherical* surface have some features in common but others in which they fundamentally differ.

Some things are doubtless axiomatically true for a spherical surface that are not for a plane surface, and *vice versa*.

Some things, also, are demonstrably true for a spherical surface that can be demonstrated not to be true for plane surfaces, and *vice versa*.

Three distinct views emerge in the answers given to the question—Is the parallel-axiom as restricted to a plane surface self evident or does it need proof?

1. Euclid assumes its axiomatic character.

2. Many geometers believe it to be a sound geometrical statement but regard it as needing proof.

3. The Non-Euclideans do not view it either as axiomatic or as demonstrable. They deserve no credit for taking this ground, since they can find no foothold except by doubting or denying that it is either self evident or capable of being proved. Their position, however, is plainly that of geometrical *agnosticism*. To call it geometrical *science* would be a misnomer.

Finally, let us go deeper and ask the question—Is the parallel-axiom considered as a geometrical statement true or false?

The Euclideans respond that it is a true geometrical statement. But if true the statement that cantradicts it must be false. Two propositions that mutually contradict each other can not both of them be true while the laws of non-contradiction and excluded-middle stand unrepealed among the statutes of logic.

The Non-Euclideans answer that they do not know whether the twelfth axiom of Euclid considered as a geometrical statement is true or false.

The geometer who refuses to confess with proud humility that he is in the same exalted condition of learned ignorance respecting geometrical fact must submit to being classed with that large majority who know some things and, also, know that they know them. The Non-Euclidians must not become discouraged, however, if they find the school houses full of geometers incorrigibly persistent in maintaining the *Hypothesis anguli recti*, and an angle sum strictly equal to two right angles.

Lobatschewsky in his theorem 19 proved that the sum of the three angles of a rectillineal triangle can not be greater than two right angles.

It is further believed that in his theorem 23 he could have demonstrated tnat the angle sum can not be less than two right angles, if he had not overlooked the important fact that the sum of two of the angles in each of the triangles constructed and admitted into the series is equal to one right angle+the acute angle, a, common to all the triangles. JOHN N. LYLE.

EDITORIALS.

"THE MONTHLY is a tonic, and an excellent one."—[E. L. Sherwood, A. M., Mississippi Normal College, Houston, Miss.]

PROF. C. A. WALDO, formerly of Greencastle, Ind., is now at the head of the Mathematical Department in Purdue University, Lafayette, Ind.

PROF. P. H. PHILBRICK is now located at Pineville, La., where he is very busily engaged in work for the Kansas City, Watkins & Gulf R. R.

PROF. O. W. ANTHONY, M. Sc., late of the Missouri Military Academy, Mexico, Mo., is now at New Windsor College, New Windsor, Md.

"I THINK the MONTHLY is doing a good work and hope it will be sustained."—[Geo. A. Osborne, S. B., Prof. of Mathematics in Mass. Inst. of Technology, and author of *Osborne's Diff. and Int. Calculus* (1891)].

"I FIND very many articles in it that are interesting and instructive, and I trust the MONTHLY may have a prosperous career."—[William J. Milne, Ph. D , LL. D., New York State Normal College, Albany, N. Y.]

PROF. E. S. LOOMIS, of Cleveland, Ohio, writes, "I find my new field of labor very agreeable, and I am now free to *think* as truth leads me out. am so constituted that I *can not* 'build to *fit* an idea, but to *find* one'."

THE American Mathematical Society held its Second Summer Meeting at Springfield, Mass., Aug. 27th and 28th, at which time and place were held also meetings of the American Association for the Advancement of Science and several other scientific societies of a national character.

WE have been obliged to issue another double number, from causes which we could not control. We hope our readers will bear with us patiently until we have completed our arrangements for the publication of the MONTHLY for next year. After that time we hope to have the MONTHLY appear regularly each month.

OUR contributors should please observe the following in reference to their contributions: (1) Write out their solutions of problems on substantial paper having a width of from 8 to 10 inches; (2) observe punctuation and capitalization; (3) write each solution on a separate slip of paper; and (4) sign your name to each solution or contribution.

"I PRIZE the MONTHLY very highly indeed. Taken all in all it is the best mathematical journal that has appeared in our country. I only wish I could find time to work out some of its excellent problems, and add a word now and then to its interesting discussions."—[Edward Brooks, A. M., Ph D., Superintendant's Office, 713 Filbert St., Philidelphia, Pa.

UNDER "*Queries & Information*," we have published two brief articles from the pen of Dr. John N. Lyle. These two articles speak for themselves. Dr. Lyle may be regarded as the greatest Anti-Non-Euclidean Geometer in America, and he has furnished many papers for publication in the MONTHLY. These we shall publish as our space permits. *Truth* has nothing to fear at the hands of any one, and if the Non-Euclidean doctrine is true, Dr. Lyle's papers will only aid in establishing it. Every great advance in science, every great discovery in nature, and every great invention has had its crowd of ridiculers; and Non-Euclidean Geometry is no exception. The Editors of the MONTHLY belong to the Non-Euclidean school of thought, even though the knowledge of that school respecting geometrical facts is an "exalted condition of learned ignorance." A school of thought represented by such men as Cayley, Sylvester, Klein, Gauss, Lambert, Lobachevsky, Halsted, Moore, and a great many others, can not be very far wrong "respecting geometric truth."

BOOKS AND PERIODICALS.

Master and Man. By Lev N. Tolstoi. Translated from the Russian by Yekaterina Alexandroona and Dr. George Bruce Halsted. Volume two of the Neomonic Series. Published at THE NEOMON, 2407 Guadalupee St., Austin, Texas.

The translation of this interesting story by Dr. Halsted, has indebted the whole literary world of the English speaking people to him. The story is very interesting from beginning to end and teaches a very useful lesson. This would have been lost to a great majority of the English people had not Dr. Halsted given them this translation. B. F. F.

The Number-System of Algebra, Treated Theoretically and Historically. By Henry B. Fine, Ph. D., Professor of Mathematics in Princeton College. 8vo. cloth, 132 pp. Price, $1.00 Boston, New York and Chicago: Leach, Shewell & Sanborn.

In this little book, are treated concisely the following subjects:

I. Theoretical. The Positive Integer; Subtraction and the Negative Integer; Division and the Fraction; The Irrational; The Imaginary and Complex Numbers: Graphical Representations of Numbers; The Fundamental Theorem of Algebra; Defined by infinite series; The Exponential and Logarithmic Functions.

II. Historical. The Primitive Numerals; Historic System of Notation; The Fraction; Origin of the Irrationals; Origin of the Negative and the Imaginary; Acceptance of the Negative, the general Irrational, and the Imaginary as Numbers: Recognition of the Purely Symbolic Nature of Algebra.

This little book should be read by every teacher of Algebra and by every student desiring to pursue the Modern Higher Mathematical Analysis. The theoretical part is very instructive and the historical part full of many interesting facts.

B. F. F.

Text-Book on Algebra through Quadratic Equations. By Joseph V. Collins, Ph. D., Professor of Mathematics in Miami University. 8vo. cloth, 468 pp.+18 pp. of *Ans.* Price, $1.00. Chicago: Albert, Scott & Co.

This book is supplied with well chosen problems and each subject is presented by numerous illustrative problems. This is a very commendable feature.

The definitions and the explanations are stated with sunlight clearness. On pages 362-366 is a discussion of the Validity of Processes of Solution of Quadratic Equations. This discussion is very appropriate and should be incorporated in all algebras. On pages 441-444 is given a correct demonstration of the theorem of Undetermined Coefficients. The demonstration of this theorem in many of our text-books on algebra is fallacious. We consider the book a credit to the subject of Elementary Algebra. B. F. F.

A Geometrical Treatment of Curves which are Isogonal Conjugate to a Straight Line with respect to a triangle. By Dr. I. J. Schwatt, University of Pennsylvania. Price, $1.00. Boston, New York, and Chicago: Leach, Shewell, and Sanborn.

The first part of the book continues the properties of Steiner's Ellipse and Kiepert's Equilateral Hyperbola, treated as isogonal conjugate curves. The writer has endeavored to give the propositions of the triangle on which the properties of the

curves are founded in a pure geometrical way. The book will prove of interest, even to such readers as are only interested in the beautiful properties of the triangle, without regard to isogonal curves. The book cannot be better recommended as it is by the eminent French athematician Vigarie, one of the greatest authorities on the Geometry of theTriangle:--

"The work has been admirably conceived, and in my belief it is the first essay of the kind that has ever been published. I do not doubt that the book will be read with the greatest interest by all those who love and cultivate the geometry of the triangle. I am convinced that the second part of the very interesting memoir will meet with the same success."—VIGARIE.

The second part will contain The Ellipse (*continued*), The Parabola, and Higher Plane Curves. B. F. F.

Geometry Tablet for Written Exercises. For use with any text-book. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Ugene Smith, Professor of Mathematics in the Michigan State Normal School. Boston and Chicago: Ginn & Co.

This tablet is prepared to assist the student in expressing his demonstrations of Original Propositions in a neat and attractive manner. Nearly all of our modern text-books on geometry are well supplied with original propositions, some books having as many as 700 originals.

In geometry classes where a year is devoted to the study, the live and progressive teacher of geometry has all of these propositions worked into the daily exercise; for geometry is now seldom taught as it was twenty years ago. A student is required to do more than merely commit to memory the demonstration of a proposition. He is now required to make the argument of the demonstration his own and until he has acquired the habit to think and to not memorize in the study of geometry, his work is that of a parrot.

The use of this tablet will enable the teacher to make careful criticisms, and the student in turn has an opportunity to study the criticisms and profit by them. I like the tablet so well that I have recommended it to my class. The tablet could be improved by making two perforations at the top so that the leaves could be bound together. B. F. F.

The Basis. A weekly journal devoted to Good Citizenship, Public Peace, Personal Security, etc. Edited by Albion W. Tourgee, Mayville, N. Y. Price, $1.50 per year. Single copy, 5c.

Judge Albion W. Tourgee's Weekly Magazine devoted to Good Citizenship as the only means by which Good Government can be surely attained, has a spicy department of Good Government Clubs, conducted by Rev. Thomas R. Slicer, of Buffalo, a department of our Women Citizens, conducted by Miss Ada C. Sweet, of Chicago, and some bright interesting pages twice a month on Boy and Girl Citizens, by Prof. W. K. Wickes, of Syracuse, besides the strong and original work of the editor in each number.

The Basis is the only journal in the world devoted to citizenship and actual government by the people. It is a thirty-two page weekly, magazine size, independent, outspoken and progressive on all subjects. Though it adheres to the principles of the National Republican party, it does not hesitate to discuss all themes with the freedom from party-bias which characterizes the works of its distinguished editor. It should be in the hands of everyone who desires to see healthful progress. B. F. F.

On The Inscription of Regular Polygons.—This is a Reprint from "The Annals of Mathematics" of an article by our valued contributor Leonard E. Dickson, M. A., Fellow in Mathematics in the University of Chicago. Those who read the author's series of articles in late numbers of the MONTHLY, treating this geometric subject without the use of the customary complex imaginary, will find special interest in the above paper, in which the author avoids the use of trigonometry also, basing his treatment solely upon algebraic and geometric principles. J. M. C.

The Educational News. A weekly Journal of Education. Edited by Dr. Albert N. Raub. Price, $1.00 per year. Published by the Educational News Co., Philadelphia, Pennsylvania.

This active and public spirited journal should be in the hands of every progressive teacher in the land. It will prevent any teacher from getting into ruts—a very common occurrence with certain classes. B. F. F.

Note on Infinite Determinants.—A dissertation presented to the faculty of the University of Mississippi for the degree of doctor of philosophy by Eugene Roberts, of Oxford, Miss. The article is based to a great extent upon von Koch's second paper (Acta Mathematica, t. 16), but in the method of treating the subject, that of considering an infinite determinant as an infinite series, the terms of which are infinite products, the author believes to be entirely new. As the result of special study of the subject Mr. Roberts has produced a very creditable paper. We are indebted to Dr. Hume, of the University of Mississippi, for a copy of the above paper in pamphlet form. J. M. C.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York City.

The special features of the October *Review of Reviews* are: Religious Jaurnalism and Journalists, by George P. Morris; the Carnegie Libraries.—Notes on a popular educational movement in the "Greater Pittsburg", by William B. Shaw; Malabeleland under the British South Africa Company, by Sir Frederick Frankland, Bart.: The Maori.—Politics and Social Life of the Native New Zelander, by Lois Becke & J. D. Fitzgerald; the Civil Service Problem in Australia, by the Sec'y of the New South Wales Royal Commission on Civil Service; the Manitoba School Question, by the Attorney-General of the Province. These articles contain numerous illustrations.

B. F. F.

The *American Journal of Mathematics* for October contains the following papers:—"On the Deformation of Thin Elastic Wires", by A. B. Basset; "Investigations in the Lunar Theory", by Ernest W. Brown; "Ueber den Sinn der Windung in den singularen Puncten einer Raumcurve", by Von Otto Staude.

Some Considerations showing the Importance of Mathematical Study. By I. J. Schwatt, Ph. D., University of Pennsylvania. This paper contains the opening address of the Mathematical Department of the third summer meeting of the American Society for the Extension of University Teaching. We are grateful to Dr. Schwatt for a copy of this inspiriting address, which is rich in illustration and forcible in presentation. J. M. C.

The Cosmopolitan. An International Illustrated Monthly Magazine.

Edited by John Brisben Walker. Price, $1.00 per year. Single Number, 10 cents.

The principal articles in the October Number of the *Cosmopolitan* are: Cuba's Struggle for Freedom, by J. Frank Clark; the Greatness of Man, by Richard Le Gallienne; State Universities, by Richard T. Ely, Ph. D., LL. D.; Mowgli Leaves the Jungle Forever, by Rudyard Kipling; Are We Old Fogies? by J. C. Ayres, Capt. U. S. A.

The Jungle Stories which have created so much interest during the spring and summer, end with this number. B. F. F.

Philosophy of the "New Law in Geometry", leading to the solution of unsolved problems. By Theodore Faber. We extend our thanks to Editor S. C. Gould for a copy of this reprint from *Notes and Queries*.

The following periodicals have been received:—The *Kansas University Quarterly*, (July); The *Monist*, (October); The *Mathematical Gazette*, (May); *Journal de Mathematiques Elementaries*, (July); *Educational Times*, (October); *L'Intermediaire des Mathematiciens*, (September); *Bulletin of the American Mathematical Society*, (July); *El Progresso Matematico*, (June); *Miscellaneous Notes and Queries*, (November).

ERRATA.

July-August Number.

p. 247, ninth line from bottom of page, for "solved" read proved.
p. 248, second line from top of page, for "psendo" read pseudo.
p. " , seventh line from top of page, for "Bell" read Ball, and
p. " , seventh line from top of page, for "Brittanica" read Britannica.
p. " , seventh line from bottom of page, for "Bered" read Berea.
p. 249, eleventh line from bottom of page, for "adopted read adapted.
p. " , fourth line from bottom of page, for "adopt" read adapt.